I'll assume you remember the definitions of continuity of a function between metric spaces:

$f : (X, \rho) \to (X', \rho')$ is continuous if for each $x \in X$ given $\varepsilon > 0$ $\exists$ $\delta > 0$ (δ may depend on x) such that

$$\rho(x, y) < \delta \Rightarrow \rho'(f(x), f(y)) < \varepsilon.$$

Similarly, we had infinite sequences $\{x_n\} \subset X$ and their convergence $x_n \to x \in X$ if given $\varepsilon > 0$ $\exists$ N (can depend on ε) such that

$$n \geq N \Rightarrow \rho(x_n, x) < \varepsilon$$

We had open sets and closed sets

$A \subset X$ is open if for each $x \in A$ $\exists$ $\delta > 0$ $\ni$ the open ball of radius δ, centered at x

$$S(x, \delta) \subset A$$

$B \subset A$ is <u>closed</u> if its complement $X - B$ is an open set in A

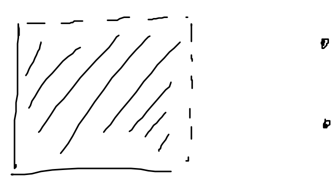
Given any subset A of X you can define its closure $[A]$.

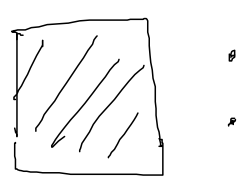
$$[A] = A \cup \{\text{limit points of } A\}$$

where $x \in X$ is a limit point of A if each open set containing x contains an elt. of A. (every point of A is a limit point of A).

For metric spaces, you just have to check that for every $\varepsilon > 0$ $S(x,\varepsilon) \cap A \neq \emptyset$

<u>Example</u>. let $A \subset \mathbb{R}^2$ be



then $[A]$ is 

Let's go back to sequences and their convergence.

Recall $(X, \rho) = \{$bi-infinite sequences of 0's and 1's$\}$

with $\rho(x, y) = \sum_{-\infty}^{\infty} \frac{|x_i - y_i|}{2^{|i|}}$

let $\{x_n\}$ be the sequence

$$(x_n)_i = \begin{cases} 0 & \text{if } i \neq n \\ 1 & \text{if } i = n \end{cases}$$

then $x_n \to 0$ in the ρ-metric.

On the other hand, if we look at $\ell^\infty(\mathbb{R}, \mathbb{Z})$

the same sequence is in $\ell^\infty(\mathbb{R}, \mathbb{Z})$

but $\rho(x_n, 0) = \sup_i |(x_n)_i - (0)_i| = 1$ for each n.

Similarly, $x_n \not\to 0$ in $\ell^p(\mathbb{R}, \mathbb{Z})$

<u>claim:</u> if $x_n \to x$ in $\ell^2(\mathbb{R}, \mathbb{N})$
then $x_n \to x$ in $\ell^3(\mathbb{R}, \mathbb{N})$.

<u>proof:</u> Since $x_n \to x$ in $\ell^2(\mathbb{R}, \mathbb{N})$ $\exists N_1$
$\ni$ $n \geq N_1 \Rightarrow \|x_n - x\|_2 < 1$.

$$\Rightarrow \sqrt{\sum_1^\infty |(x_n)_i - x_i|^2} < 1$$

$\Rightarrow$ for each i, $|(x_n)_i - x_i| < 1$

Now, let $\varepsilon > 0$. Since $x_n \to x$ in ℓ^2
$\exists N_\varepsilon \ni n \geq N_\varepsilon \Rightarrow \|x_n - x\|_2 < \varepsilon$.

Take $\tilde{N} = \max\{N_\varepsilon, N_1\}$. then for $n \geq \tilde{N}$
we have $\|x_n - x\|_2 < \varepsilon$ and $|(x_n)_i - x_i| < 1$

$$\sum_1^\infty |(x_n)_i - x_i|^3 \leq \sum_1^\infty |(x_n)_i - x_i|^2$$ because $|(x_n)_i - x_i| < 1$

$$\leq \varepsilon^2$$ because $\|x_n - x\|_2 < \varepsilon$

$\Rightarrow \|x_n - x\|_3 < \varepsilon^{2/3}$. We can take $\varepsilon^{2/3}$ as small as we want $\Rightarrow x_n \to x$ in ℓ^3

<u>theorem</u>: if $x_n \to x$ in $\ell^p(\mathbb{R}, \mathbb{N})$
then $x_n \to x$ in $\ell^q(\mathbb{R}, \mathbb{N})$ $\forall\, q \in (p, \infty]$

<u>proof</u>: repeat the argument I did for ℓ^2 & ℓ^3.

<u>fact</u>: just because $x_n \to x$ in $\ell^p(\mathbb{R}, \mathbb{N})$ doesn't imply $x_n \to x$ in ℓ^q for $q < p$. (ℓ^2 convergence $\not\Rightarrow$ ℓ^1 conv.)
why? just take a sequence x_n that doesn't live in ℓ^q.

We have a similar convergence theorem for $L^p([a,b])$ if the interval is bounded. (And in general for $L^p(\Omega)$ if Ω is a bounded measurable subset of $\mathbb{R}^n$.)

This will follow directly from Hölder's Integral inequality.

Recall that if $f \in L^p([a,b])$ then $f \in L^q([a,b])$ if $q \in [1,p)$

AS LONG AS THE INTERVAL $[a,b]$ IS BOUNDED!

Specifically, we showed

$$\|f\|_{L^q[a,b]} \le |b-a|^{\frac{p-q}{pq}} \|f\|_{L^p([a,b])}$$

if $q \in [1,p)$

<u>theorem</u>: if $f_n \to f$ in $L^p([a,b])$ where $[a,b]$ is a bounded interval then $f_n \to f$ in $L^q([a,b])$ for $q \in [1,p)$

<u>proof</u>: Since $f_n \to f$ in L^p, we know that given $\varepsilon > 0$ $\exists N_\varepsilon \ni n \geq N_\varepsilon \Rightarrow$

$$\|f_n - f\|_p < \varepsilon \Rightarrow \|f_n - f\|_q < \varepsilon |b-a|^{\frac{p-q}{pq}}$$

can make RHS as small as we need. //

CONVERGENCE IN $L^1([0,1])$ DOES NOT IMPLY CONVERGENCE IN $L^2([0,1])$

Example:

let $f_n(t) = \begin{cases} 0 & \text{if } 0 \le x < 1/n \\ \frac{1}{\sqrt{x}} & \text{if } \frac{1}{n} \le x < 1 \end{cases}$

let $g(x) = 0$ for $x \in [0,1]$

then $\|f_n - g\|_{L^1} = \int_{1/n}^{1} \frac{1}{\sqrt{x}}\,dx = 2\left(1 - \frac{1}{\sqrt{n}}\right)$

$$\|f_n - g\|_{L^2} = \sqrt{\int_{1/n}^{1} \left(\frac{1}{\sqrt{x}}\right)^2 dx} = \sqrt{\ln(n)}$$

so f_n is getting to distance 1 from g in the L^1 metric, but getting infinitely far away in L^2 metric.

if $h_n = \frac{f_n}{\sqrt{\ln(n)}}$ then $h_n \to g$ in l^1

$h_n \not\to g$ in l^2

And, as you'd suspect, infinite domains cause other troubles.

ex: $f_n(x) = \begin{cases} \frac{1}{n} & n < x < 2n \\ 0 & \text{otherwise} \end{cases}$

then $\int_0^\infty |f_n(x)|\,dx = 1$

$\int_0^\infty |f_n(x)|^2\,dx = \frac{1}{n}$

so $f_n \to 0$ in $L^2([0,\infty))$

but $f_n \not\to 0$ in $L^1([0,\infty))$

Note: this example has $f_n \to 0$ in $L^p([0,\infty))$ for any $p > 1$

Note: You can similarly make an example $\ni$ $\|f_n\|_p = 1$ but $\|f_n\|_{L^q} \to 0$ for any $q > p$.