Recall $\ell^p(\mathbb{R}, \mathbb{N})$. This is the space of infinite sequences $x$, such that

$$\sqrt[p]{\sum_1^\infty |x_j|^p} < \infty.$$

From last time, know that

$$\rho(x,y) = \sqrt[p]{\sum_1^\infty |x_j - y_j|^p} =: \|x-y\|_p$$

is a metric if $p \in [1, \infty)$. Similarly,

$$\|x-y\|_\infty = \sup_{1 \le j \le \infty} |x_j - y_j| \quad \text{is a metric.}$$

What are elements of $\ell^p(\mathbb{R}, \mathbb{N})$?

$$x = \{\lambda^j\} \in \ell^p \quad \text{if } p \in (1, \infty]$$
$$\text{if } |\lambda| < 1$$

since $$\sum_1^\infty |\lambda^j| = \sum_1^\infty |\lambda|^j = \frac{|\lambda|}{1-|\lambda|} \quad \text{if } |\lambda| < 1$$

$$\Rightarrow x \in \ell^1(\mathbb{R}, \mathbb{N})$$

and $$\sqrt[p]{\sum_1^\infty |\lambda^j|^p} = \sqrt[p]{\frac{|\lambda|^p}{1-|\lambda|^p}} \quad \text{if } |\lambda|^p < 1$$

$\Rightarrow x \in \ell^p(\mathbb{R}, \mathbb{N})$ as long as $p > 1$

and $\sup_{1 \le j < \infty} |\lambda^j| < \infty$ if $|\lambda| < 1 \Rightarrow x \in \ell^\infty(\mathbb{R}, \mathbb{N})$

If $x = \{ \frac{1}{\sqrt{j}} \}$

then you can check that

$x \in l^p(\mathbb{R}, \mathbb{N})$ for $p \in [2, \infty]$

Q: if $p > q$ and $x \in l^q(\mathbb{R}, \mathbb{N})$, does this imply $x \in l^p(\mathbb{R}, \mathbb{N})$? Does this imply $x \in l^\infty(\mathbb{R}, \mathbb{N})$?

First, $l^\infty$. yes because if $x \in l^q$ then

$$\sqrt[q]{\sum_1^\infty |x_j|^q} < \infty \Rightarrow \lim_{j \to \infty} |x_j|^q = 0$$

$$\Rightarrow \{ |x_j| \} \text{ is bounded} \Rightarrow x \in l^\infty(\mathbb{R}, \mathbb{N}).$$

Now $l^p$. yes too. Why? Again,

$$x \in l^q \Rightarrow |x_j| \leq 1 \quad \forall j \geq N \text{ for some } N.$$

let $S_n = \sum_1^n |x_j|^p$ we want to bound

$S_n \leq M$ since then $\lim_{n \to \infty} S_n$ exists and

$$\sum_1^\infty |x_j|^p < \infty.$$

$$S_n = \sum_1^n |x_j|^p = \sum_1^N |x_j|^p + \sum_{N+1}^n |x_j|^p$$

$$\leq \sum_1^N |x_j|^p + \sum_{N+1}^n |x_j|^q$$ bec. $p > q$ and $|x_j| \leq 1$ for $j \geq N$

$$\leq C + \tilde{S}_n \quad \text{where} \quad \tilde{S}_n = \sum_1^n |x_j|^q$$

Since $x \in \ell^q$, we know $\tilde{S}_n$ is bounded $\Rightarrow S_n$ is bounded $\Rightarrow x \in \ell^p$ as desired. //

Okay, the $\ell^p$ spaces are understandable, if you're comfortable w/ infinite series. And one thing that we've lucky with is

$$\ell^q(\mathbb{R}, \mathbb{N}) \subset \ell^p(\mathbb{R}, \mathbb{N})$$

if $q < p$. (We just showed this.) Also, we know it's not a strict inclusion. $\exists\ x \in \ell^p$ that aren't in $\ell^q$.

Here's a trickier space

$$L^p(\mathbb{R}) = \left\{ f \;\middle|\; \sqrt[p]{\int_{-\infty}^{\infty} |f(x)|^p \, dx} < \infty \right\}$$

$$L^\infty(\mathbb{R}) = \left\{ f \;\middle|\; \sup_{x \in \mathbb{R}} |f(x)| < \infty \right\}.$$

Actually, you can define $L^p(X)$ for any subset of $\mathbb{R}$. So let's start with a simpler case

$$L^p([0,1]) = \left\{ f \;\middle|\; \sqrt[p]{\int_0^1 |f(x)|^p} < \infty \right\}$$

$$L^\infty([0,1]) = \left\{ f \;\middle|\; \sup_{x \in [0,1]} |f(x)| < \infty \right\}$$

Here, when I write $\int_0^1 |f(x)|^p$ I mean the Riemann Integral from Calculus.

For convenience, you can imagine that we're thinking about functions with finitely many discontinuities.

ex:

$$f(x) = \begin{cases} 1/\sqrt{x} & x \in (0,1] \\ 3 & x = 0 \end{cases}$$

then $f \notin L^\infty([0,1])$ check $L^p$

$$\int_0^1 |f(x)|^p dx = \int_0^1 \left(\frac{1}{\sqrt{x}}\right)^p dx = \int_0^1 x^{-p/2}\, dx$$

$$= \left.\frac{x^{1-p/2}}{1-p/2}\right|_0^1 \qquad \text{if } p \neq 2$$

so $\int_0^1 |f(x)|^p dx < \infty$ if $p < 2$

if $p=2$ then $\int_0^1 \left(\frac{1}{\sqrt{x}}\right)^p dx = \int_0^1 \frac{1}{x}\, dx = \ln x \Big|_0^1 \not< \infty$

$\Rightarrow f(x) \in L^p([0,1])$ for $p \in [1,2)$

Note: I'm not considering $p<1$ because $\|\cdot\|_p$ isn't a metric for $p<1$

Note: All of the above were really improper integrals: $\lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 \left(\frac{1}{\sqrt{x}}\right)^p dx$

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For $l^p(\mathbb{R}, \mathbb{N})$, we could prove some inclusions:

$$l^p(\mathbb{R}, \mathbb{N}) \subseteq l^q(\mathbb{R}, \mathbb{N}) \text{ if } q > p$$

What inclusions do we have for $L^p([0,1])$?

__theorem:__ if $f \in L^q([0,1])$ then $f \in L^p([0,1])$ for all $p \in [1, q)$

(so the inclusion is $L^q([0,1]) \subseteq L^p([0,1])$ if $q > p$. The opposite of $l^p$ and $l^q$.)

__proof:__ take $p \in [1, q)$

then
$$\int_0^1 |f(x)|^p \, dx = \int_0^1 1 \cdot |f(x)|^p \, dx$$

$$\leq \sqrt[\tilde{p}]{\int_0^1 1^{\tilde{p}} \, dx} \; \sqrt[\tilde{q}]{\int_0^1 |f(x)|^{p\tilde{q}}}$$

for any $\tilde{p}, \tilde{q} \in (1, \infty)$ with $\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = 1$

take $\tilde{q} = \frac{q}{p} > 1$ (by assumption)

then $\tilde{p} = \frac{q}{q-p}$

So $\int_0^1 |f(x)|^p dx \le (1)^{\frac{q-p}{p}} \sqrt[\tilde{q}]{\int_0^1 |f(x)|^q dx} < \infty$

because $f \in L^q([0,1])$.

Note: I used Hölder's Integral inequality here. It's a HW problem to prove it's true for Riemann Integrable functions.

Note: if $L^p([a,b])$ then we still have

$$f \in L^q([a,b]) \Rightarrow f \in L^p([a,b]) \text{ for } p \in [1,q).$$

the above proof goes through with

$$\int_a^b |f(x)|^p dx \le |b-a|^{\frac{q-p}{p}} \sqrt[\tilde{q}]{\int_a^b |f(x)|^q dx}$$

Also, let's unravel the above a little more...

$$\int_a^b |f(x)|^p dx \le |b-a|^{\frac{q-p}{p}} \left(\int_a^b |f(x)|^q\right)^{1/\tilde{q}}$$

$$= |b-a|^{\frac{q-p}{p}} \left(\int_a^b |f(x)|^q\right)^{p/q}$$

$$\|f\|_p \le |b-a|^{\frac{q-p}{pq}} \|f\|_q$$

Notice that as the interfval gets larger, the right hand side goes to infinity!

So we don't have inclusions like that for $L^p([0,\infty))$. Or at least not by that proof...

Let's look at $L^p([0,\infty))$.

take $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & x > 0 \\ 3 & x = 0 \end{cases}$

then $$\int_0^\infty |f(x)|^p dx = \int_0^8 |f(x)|^p dx + \int_8^\infty |f(x)|^p dx$$

(the 8 is arbitrary, but to make sense of this improper integral we have to cut it somewhere...)

$$= \int_0^8 x^{-p/2} dx + \int_8^\infty x^{-p/2} dx$$

$$= \left. \frac{x^{1-p/2}}{1-p/2} \right|_0^8 + \left. \frac{x^{1-p/2}}{1-p/2} \right|_8^\infty \qquad \text{if } p \neq 2$$

So if you choose $p$ so that

$\int_0^8$ is finite, you get $\int_8^\infty$ infinite

and vica versa. $\Rightarrow f \notin L^p([0,\infty))$ for **any** $p$.

$$f(x) = \begin{cases} \frac{1}{\sqrt[3]{x}} & x \in (0,2] \\ \frac{1}{x^2} & x \in (2,\infty) \\ 8 & x = 0 \end{cases}$$

has $f(x) \in L^p([0,\infty))$ for $1 \leq p < 3$.