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THE STONE-WEIERSTRASS THEOREM

7.26 Theorem *If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that*

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If f is real, the P_n may be taken real.

This is the form in which the theorem was originally discovered by Weierstrass.

Proof We may assume, without loss of generality, that $[a, b] = [0, 1]$. We may also assume that $f(0) = f(1) = 0$. For if the theorem is proved for this case, consider

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \qquad (0 \le x \le 1).$$

Here $g(0) = g(1) = 0$, and if g can be obtained as the limit of a uniformly convergent sequence of polynomials, it is clear that the same is true for f, since $f - g$ is a polynomial.

Furthermore, we define $f(x)$ to be zero for x outside $[0, 1]$. Then f is uniformly continuous on the whole line.

We put

(47)
$$Q_n(x) = c_n(1 - x^2)^n \qquad (n = 1, 2, 3, \ldots),$$

where c_n is chosen so that

(48)
$$\int_{-1}^{1} Q_n(x)\, dx = 1 \qquad (n = 1, 2, 3, \ldots).$$

We need some information about the order of magnitude of c_n. Since

$$\begin{aligned}\int_{-1}^{1} (1 - x^2)^n\, dx = 2\int_0^1 (1 - x^2)^n\, dx &\ge 2\int_0^{1/\sqrt{n}} (1 - x^2)^n\, dx \\ &\ge 2\int_0^{1/\sqrt{n}} (1 - nx^2)\, dx \\ &= \frac{4}{3\sqrt{n}} \\ &> \frac{1}{\sqrt{n}},\end{aligned}$$

it follows from (48) that

(49)
$$c_n < \sqrt{n}.$$

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The inequality $(1 - x^2)^n \geq 1 - nx^2$ which we used above is easily shown to be true by considering the function

$$(1 - x^2)^n - 1 + nx^2$$

which is zero at $x = 0$ and whose derivative is positive in $(0, 1)$.

For any $\delta > 0$, (49) implies

(50) $$Q_n(x) \leq \sqrt{n}\,(1 - \delta^2)^n \qquad (\delta \leq |x| \leq 1),$$

so that $Q_n \to 0$ uniformly in $\delta \leq |x| \leq 1$.

Now set

(51) $$P_n(x) = \int_{-1}^{1} f(x + t)Q_n(t)\,dt \qquad (0 \leq x \leq 1).$$

Our assumptions about f show, by a simple change of variable, that

$$P_n(x) = \int_{-x}^{1-x} f(x + t)Q_n(t)\,dt = \int_0^1 f(t)Q_n(t - x)\,dt,$$

and the last integral is clearly a polynomial in x. Thus $\{P_n\}$ is a sequence of polynomials, which are real if f is real.

Given $\varepsilon > 0$, we choose $\delta > 0$ such that $|y - x| < \delta$ implies

$$|f(y) - f(x)| < \frac{\varepsilon}{2}.$$

Let $M = \sup |f(x)|$. Using (48), (50), and the fact that $Q_n(x) \geq 0$, we see that for $0 \leq x \leq 1$,

$$\begin{aligned}
|P_n(x) - f(x)| &= \left| \int_{-1}^{1} [f(x + t) - f(x)]Q_n(t)\,dt \right| \\
&\leq \int_{-1}^{1} |f(x + t) - f(x)|\,Q_n(t)\,dt \\
&\leq 2M \int_{-1}^{-\delta} Q_n(t)\,dt + \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t)\,dt + 2M \int_{\delta}^{1} Q_n(t)\,dt \\
&\leq 4M\sqrt{n}\,(1 - \delta^2)^n + \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

for all large enough n, which proves the theorem.

It is instructive to sketch the graphs of Q_n for a few values of n; also, note that we needed uniform continuity of f to deduce uniform convergence of $\{P_n\}$.

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In the proof of Theorem 7.32 we shall not need the full strength of Theorem 7.26, but only the following special case, which we state as a corollary.

7.27 Corollary *For every interval $[-a, a]$ there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that*

$$\lim_{n \to \infty} P_n(x) = |x|$$

uniformly on $[-a, a]$.

Proof By Theorem 7.26, there exists a sequence $\{P_n^*\}$ of real polynomials which converges to $|x|$ uniformly on $[-a, a]$. In particular, $P_n^*(0) \to 0$ as $n \to \infty$. The polynomials

$$P_n(x) = P_n^*(x) - P_n^*(0) \qquad (n = 1, 2, 3, \ldots)$$

have desired properties.

We shall now isolate those properties of the polynomials which make the Weierstrass theorem possible.

7.28 Definition A family $\mathscr{A}$ of complex functions defined on a set E is said to be an *algebra* if (i) $f + g \in \mathscr{A}$, (ii) $fg \in \mathscr{A}$, and (iii) $cf \in \mathscr{A}$ for all $f \in \mathscr{A}$, $g \in \mathscr{A}$ and for all complex constants c, that is, if $\mathscr{A}$ is closed under addition, multiplication, and scalar multiplication. We shall also have to consider algebras of real functions; in this case, (iii) is of course only required to hold for all real c.

If $\mathscr{A}$ has the property that $f \in \mathscr{A}$ whenever $f_n \in \mathscr{A}$ $(n = 1, 2, 3, \ldots)$ and $f_n \to f$ uniformly on E, then $\mathscr{A}$ is said to be *uniformly closed.*

Let $\mathscr{B}$ be the set of all functions which are limits of uniformly convergent sequences of members of $\mathscr{A}$. Then $\mathscr{B}$ is called the *uniform closure* of $\mathscr{A}$. (See Definition 7.14.)

For example, the set of all polynomials is an algebra, and the Weierstrass theorem may be stated by saying that the set of continuous functions on $[a, b]$ is the uniform closure of the set of polynomials on $[a, b]$.

7.29 Theorem *Let $\mathscr{B}$ be the uniform closure of an algebra $\mathscr{A}$ of bounded functions. Then $\mathscr{B}$ is a uniformly closed algebra.*

Proof If $f \in \mathscr{B}$ and $g \in \mathscr{B}$, there exist uniformly convergent sequences $\{f_n\}, \{g_n\}$ such that $f_n \to f$, $g_n \to g$ and $f_n \in \mathscr{A}$, $g_n \in \mathscr{A}$. Since we are dealing with bounded functions, it is easy to show that

$$f_n + g_n \to f + g, \qquad f_n g_n \to fg, \qquad cf_n \to cf,$$

where c is any constant, the convergence being uniform in each case.

Hence $f + g \in \mathscr{B}$, $fg \in \mathscr{B}$, and $cf \in \mathscr{B}$, so that $\mathscr{B}$ is an algebra.

By Theorem 2.27, $\mathscr{B}$ is (uniformly) closed.

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7.30 Definition Let $\mathscr{A}$ be a family of functions on a set E. Then $\mathscr{A}$ is said to *separate points* on E if to every pair of distincts point $x_1, x_2 \in E$ there corresponds a function $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.

If to each $x \in E$ there corresponds a function $g \in \mathscr{A}$ such that $g(x) \neq 0$, we say that $\mathscr{A}$ *vanishes at no point of E.*

The algebra of all polynomials in one variable clearly has these properties on R^1. An example of an algebra which does not separate points is the set of all even polynomials, say on $[-1, 1]$, since $f(-x) = f(x)$ for every even function f.

The following theorem will illustrate these concepts further.

7.31 Theorem *Suppose $\mathscr{A}$ is an algebra of functions on a set E, $\mathscr{A}$ separates points on E, and $\mathscr{A}$ vanishes at no point of E. Suppose x_1, x_2 are distinct points of E, and c_1, c_2 are constants (real if $\mathscr{A}$ is a real algebra). Then $\mathscr{A}$ contains a function f such that*

$$f(x_1) = c_1, \qquad f(x_2) = c_2.$$

Proof The assumptions show that $\mathscr{A}$ contains functions g, h, and k such that

$$g(x_1) \neq g(x_2), \qquad h(x_1) \neq 0, \qquad k(x_2) \neq 0.$$

Put

$$u = gk - g(x_1)k, \qquad v = gh - g(x_2)h.$$

Then $u \in \mathscr{A}$, $v \in \mathscr{A}$, $u(x_1) = v(x_2) = 0$, $u(x_2) \neq 0$, and $v(x_1) \neq 0$. Therefore

$$f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$$

has the desired properties.

We now have all the material needed for Stone's generalization of the Weierstrass theorem.

7.32 Theorem *Let $\mathscr{A}$ be an algebra of real continuous functions on a compact set K. If $\mathscr{A}$ separates points on K and if $\mathscr{A}$ vanishes at no point of K, then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.*

We shall divide the proof into four steps.

STEP 1 *If $f \in \mathscr{B}$, then $|f| \in \mathscr{B}$.*

Proof Let

$$a = \sup |f(x)| \qquad (x \in K) \tag{52}$$

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and let $\varepsilon > 0$ be given. By Corollary 7.27 there exist real numbers $c_1, \ldots, c_n$ such that

(53) $$\left| \sum_{i=1}^{n} c_i y^i - |y| \right| < \varepsilon \qquad (-a \le y \le a).$$

Since $\mathscr{B}$ is an algebra, the function

$$g = \sum_{i=1}^{n} c_i f^i$$

is a member of $\mathscr{B}$. By (52) and (53), we have

$$\big| g(x) - |f(x)| \big| < \varepsilon \qquad (x \in K).$$

Since $\mathscr{B}$ is uniformly closed, this shows that $|f| \in \mathscr{B}$.

STEP 2 *If $f \in \mathscr{B}$ and $g \in \mathscr{B}$, then $\max(f, g) \in \mathscr{B}$ and $\min(f, g) \in \mathscr{B}$.*

By $\max(f, g)$ we mean the function h defined by

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \ge g(x), \\ g(x) & \text{if } f(x) < g(x), \end{cases}$$

and $\min(f, g)$ is defined likewise.

Proof Step 2 follows from step 1 and the identities

$$\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2},$$

$$\min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

By iteration, the result can of course be extended to any finite set of functions: If $f_1, \ldots, f_n \in \mathscr{B}$, then $\max(f_1, \ldots, f_n) \in \mathscr{B}$, and

$$\min(f_1, \ldots, f_n) \in \mathscr{B}.$$

STEP 3 *Given a real function f, continuous on K, a point $x \in K$, and $\varepsilon > 0$, there exists a function $g_x \in \mathscr{B}$ such that $g_x(x) = f(x)$ and*

(54) $$g_x(t) > f(t) - \varepsilon \qquad (t \in K).$$

Proof Since $\mathscr{A} \subset \mathscr{B}$ and $\mathscr{A}$ satisfies the hypotheses of Theorem 7.31 so does $\mathscr{B}$. Hence, for every $y \in K$, we can find a function $h_y \in \mathscr{B}$ such that

(55) $$h_y(x) = f(x), \qquad h_y(y) = f(y).$$

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By the continuity of h_y there exists an open set J_y, containing y, such that

(56) $$h_y(t) > f(t) - \varepsilon \qquad (t \in J_y).$$

Since K is compact, there is a finite set of points $y_1, \ldots, y_n$ such that

(57) $$K \subset J_{y_1} \cup \cdots \cup J_{y_n}.$$

Put

$$g_x = \max(h_{y_1}, \ldots, h_{y_n}).$$

By step 2, $g \in \mathscr{B}$, and the relations (55) to (57) show that g_x has the other required properties.

STEP 4 *Given a real function f, continuous on K, and $\varepsilon > 0$, there exists a function $h \in \mathscr{B}$ such that*

(58) $$|h(x) - f(x)| < \varepsilon \qquad (x \in K).$$

Since $\mathscr{B}$ is uniformly closed, this statement is equivalent to the conclusion of the theorem.

Proof Let us consider the functions g_x, for each $x \in K$, constructed in step 3. By the continuity of g_x, there exist open sets V_x containing x, such that

(59) $$g_x(t) < f(t) + \varepsilon \qquad (t \in V_x).$$

Since K is compact, there exists a finite set of points $x_1, \ldots, x_m$ such that

(60) $$K \subset V_{x_1} \cup \cdots \cup V_{x_m}.$$

Put

$$h = \min(g_{x_1}, \ldots, g_{x_m}).$$

By step 2, $h \in \mathscr{B}$, and (54) implies

(61) $$h(t) > f(t) - \varepsilon \qquad (t \in K),$$

whereas (59) and (60) imply

(62) $$h(t) < f(t) + \varepsilon \qquad (t \in K).$$

Finally, (58) follows from (61) and (62).

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Theorem 7.32 does not hold for complex algebras. A counterexample is given in Exercise 21. However, the conclusion of the theorem does hold, even for complex algebras, if an extra condition is imposed on $\mathscr{A}$, namely, that $\mathscr{A}$ be *self-adjoint*. This means that for every $f \in \mathscr{A}$ its complex conjugate $\bar{f}$ must also belong to $\mathscr{A}$; $\bar{f}$ is defined by $\bar{f}(x) = \overline{f(x)}$.

7.33 Theorem *Suppose $\mathscr{A}$ is a self-adjoint algebra of complex continuous functions on a compact set K, $\mathscr{A}$ separates points on K, and $\mathscr{A}$ vanishes at no point of K. Then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on K. In other words, $\mathscr{A}$ is dense $\mathscr{C}(K)$.*

Proof Let $\mathscr{A}_R$ be the set of all real functions on K which belong to $\mathscr{A}$.

If $f \in \mathscr{A}$ and $f = u + iv$, with u, v real, then $2u = f + \bar{f}$, and since $\mathscr{A}$ is self-adjoint, we see that $u \in \mathscr{A}_R$. If $x_1 \neq x_2$, there exists $f \in \mathscr{A}$ such that $f(x_1) = 1, f(x_2) = 0$; hence $0 = u(x_2) \neq u(x_1) = 1$, which shows that $\mathscr{A}_R$ separates points on K. If $x \in K$, then $g(x) \neq 0$ for some $g \in \mathscr{A}$, and there is a complex number λ such that $\lambda g(x) > 0$; if $f = \lambda g, f = u + iv$, it follows that $u(x) > 0$; hence $\mathscr{A}_R$ vanishes at no point of K.

Thus $\mathscr{A}_R$ satisfies the hypotheses of Theorem 7.32. It follows that every real continuous function on K lies in the uniform closure of $\mathscr{A}_R$, hence lies in $\mathscr{B}$. If f is a complex continuous function on K, $f = u + iv$, then $u \in \mathscr{B}$, $v \in \mathscr{B}$, hence $f \in \mathscr{B}$. This completes the proof.

EXERCISES

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.
2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.
3. Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).
4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?