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c. Show that $\beta(X)$ is unique in the sense that if $Z$ is another space with the same properties, then there is a homeomorphism $\varphi$ of $Z$ with $\beta(X)$ such that $\varphi(x) = x$ for all $x \in X$.

**40.** Let $X$ and $Y$ be the spaces in Problems 11 and 12. Show that $\beta(X) = Y$.

**41.** Let **N** be the set of natural numbers. Discuss $\beta(\mathbf{N})$. Show that a sequence from **N** converges in $\beta(\mathbf{N})$ if and only if it converges in **N**. Hence $\beta(\mathbf{N})$ is compact but not sequentially compact.

## 9 The Stone–Weierstrass Theorem

Let $X$ be a compact Hausdorff space. We denote by $C(X)$ the set of all continuous real-valued functions on $X$. Since $X$ is normal, it follows from Urysohn's Lemma that there are enough functions in $C(X)$ to separate points; that is, given two distinct points $x$ and $y$ in $X$, we can find an $f$ in $C(X)$ such that $f(x) \neq f(y)$. The set $C(X)$ is a linear space, since any constant multiple of a continuous real-valued function is continuous and the sum of two continuous functions is continuous. The space $C(X)$ becomes a normed linear space if we define $\|f\| = \max |f(x)|$, and a metric space if we set $\rho(f, g) = \|f - g\|$. As a metric space $C(X)$ is complete.

The space $C(X)$ has also a ring structure: The product $fg$ of two functions $f$ and $g$ in $C(X)$ is again in $C(X)$. A linear space $A$ of functions in $C(X)$ is called an **algebra** if the product of any two elements in $A$ is again in $A$. Thus $A$ is an algebra if for any two functions $f$ and $g$ in $A$ and any real numbers $a$ and $b$ we have $af + bg$ in $A$ and $fg$ in $A$. A family $A$ of functions on $X$ is said to separate points if given distinct points $x$ and $y$ of $X$ there is an $f$ in $A$ such that $f(x) \neq f(y)$. In the present section we study the closed subalgebras of $C(X)$ and prove that if $A$ is a subalgebra of $C(X)$ that separates points, contains the constant functions, and is closed, then $A = C(X)$.

The space $C(X)$ also has a lattice structure: If $f$ and $g$ are in $C(X)$, so is the function $f \wedge g$ defined by $(f \wedge g)(x) = \min [f(x), g(x)]$ and the function $f \vee g$ defined by $(f \vee g)(x) = \max [f(x), g(x)]$. A subset $L$ of $C(X)$ is called a **lattice** if for every pair of functions $f$ and $g$ in $L$ we also have $f \vee g$ and $f \wedge g$ in $L$. It is convenient to investigate subalgebras of $C(X)$ by first investigating lattices of functions. The following proposition can be thought of as a generalization of the Dini Theorem:

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**29. Proposition:** *Let $L$ be a lattice of continuous real-valued functions on a compact space $X$, and suppose that the function $h$ defined by*

$$h(x) = \inf_{f \in L} f(x)$$

*is continuous. Then, given $\epsilon > 0$, there is a $g$ in $L$ such that $0 \leq g(x) - h(x) < \epsilon$ for all $x$ in $X$.*

***Proof:*** For each $x$ in $X$ there is a function $f_x$ in $L$ such that $f_x(x) < h(x) + \epsilon/3$. Since $f_x$ and $h$ are continuous, there is an open set $O_x$ containing $x$ such that

$$|f_x(y) - f_x(x)| < \frac{\epsilon}{3} \quad \text{and} \quad |h(y) - h(x)| < \frac{\epsilon}{3}$$

for all $y \in O_x$. Hence $f_x(y) - h(y) < \epsilon$ for all $y$ in $O_x$. Now the sets $O_x$ cover $X$, and by compactness there are a finite number of them, say $\{O_{x_1}, \ldots, O_{x_n}\}$, which cover $X$. Let $g = f_{x_1} \wedge f_{x_2} \wedge \cdots \wedge f_{x_n}$. Then $g \in L$, and given $y$ in $X$ we may choose $i$ so that $y \in O_{x_i}$, whence

$$g(y) - h(y) \leq f_{x_i}(y) - h(y) < \epsilon. \quad ∎$$

**30. Proposition:** *Let $X$ be a compact space and $L$ a lattice of continuous real-valued functions on $X$ with the following properties:*

i. *$L$ separates points; that is, if $x \neq y$, there is an $f \in L$ with $f(x) \neq f(y)$.*
ii. *If $f \in L$, and $c$ is any real number, then $cf$ and $c + f$ also belong to $L$.*

*Then given any continuous real-valued function $h$ on $X$ and any $\epsilon > 0$, there is a function $g \in L$ such that for all $x \in X$*

$$0 \leq g(x) - h(x) < \epsilon.$$

Before proving the proposition, we first establish two lemmas.

**31. Lemma:** *Let $L$ be a family of real-valued functions on a compact space $X$ that satisfies properties (i) and (ii) of Proposition 30. Then given any two real numbers $a$ and $b$ and any pair $x$ and $y$ of distinct points of $X$, there is an $f \in L$ such that $f(x) = a$ and $f(y) = b$.*

***Proof:*** Let $g$ be a function in $L$ such that $g(x) \neq g(y)$. Let

$$f = \frac{a - b}{g(x) - g(y)}\, g + \frac{bg(x) - ag(y)}{g(x) - g(y)}.$$

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Then $f \varepsilon L$, by property (ii), and $f(x) = a, f(y) = b$. ▮

**32. Lemma:** *Let $L$ be as in Proposition 30, $a$ and $b$ real numbers with $a \leq b$, $F$ a closed subset of $X$, and $p$ a point not in $F$. Then there is a function $f$ in $L$ such that $f \geq a, f(p) = a$, and $f(x) > b$ for all $x \varepsilon F$.*

***Proof:*** By Lemma 31 we can choose, for each $x \varepsilon F$, a function $f_x$ such that $f_x(p) = a$ and $f_x(x) = b + 1$. Let $O_x = \{y: f_x(y) > b\}$. Then the sets $\{O_x\}$ cover $F$, and since $F$ is compact, there are a finite number $\{O_{x_1}, \ldots, O_{x_n}\}$ that cover $F$. Let $f = f_{x_1} \vee \cdots \vee f_{x_n}$. Then $f \varepsilon L, f(p) = a$, and $f > b$ on $F$. If we replace $f$ by $f \vee a$, then we also have $f \geq a$ on $X$. ▮

***Proof of Proposition 30:*** Since $L$ is nonempty, it follows from (ii) that the constant functions belong to $L$. Given $g \varepsilon C(X)$, let $L' = \{f: f \varepsilon L \text{ and } f \geq g\}$. Proposition 30 will follow from Proposition 29 if we can show that for each $p \varepsilon X$ we have $g(p) = \inf f(p), f \varepsilon L'$. Choose a positive real number $\eta$. Since $g$ is continuous, the set

$$F = \{x: g(x) \geq g(p) + \eta\}$$

is closed. Since $X$ is compact, $g$ is bounded on $X$, say by $M$. By Lemma 32 we can find a function $f \varepsilon L$ such that $f \geq g(p) + \eta$, $f(p) = g(p) + \eta$, and $f(x) > M$ on $F$. Since $g < g(p) + \eta$ on $\tilde{F}$, we have $g < f$ on $X$. Thus $f \varepsilon L'$, and $f(p) \leq g(p) + \eta$. Since $\eta$ was an arbitrary positive number, we have $g(p) = \inf f(p), f \varepsilon L'$. ▮

**33. Lemma:** *Given $\epsilon > 0$, there is a polynomial $P$ in one variable such that for all $s \varepsilon [-1, 1]$ we have $|P(s) - |s|| < \epsilon$.*

***Proof:*** Let $\sum_{n=0}^{\infty} c_n t^n$ be the binomial series for $(1 - t)^{1/2}$. This series converges uniformly for $t$ in the interval $[0, 1]$. Hence, given $\epsilon > 0$, we can choose $N$ so that for all $t \varepsilon [0, 1]$ we have

$$|(1 - t)^{1/2} - Q_N(t)| < \epsilon,$$

where $Q_N = \sum_{n=0}^{N} c_n t^n$. Let $P(s) = Q_N(1 - s^2)$. Then $P$ is a polynomial in $s$, and $||s| - P(s)| < \epsilon$ for $s \varepsilon [-1, 1]$. ▮

**34. Theorem (Stone–Weierstrass):** *Let $X$ be a compact space and $A$ an algebra of continuous real-valued functions on $X$ that separates*

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*the points of $X$ and contains the constant functions. Then given any continuous real-valued function $f$ on $X$ and any $\epsilon > 0$ there is a function $g$ in $A$ such that for all $x$ in $X$ we have $|g(x) - f(x)| < \epsilon$. In other words, $A$ is a dense subset of $C(X)$.*

***Proof:*** Let $\bar{A}$ denote the closure of $A$ considered as a subset of $C(X)$. Thus $\bar{A}$ consists of those functions on $X$ that are uniform limits of sequences of functions from $A$. It is easy to verify that $\bar{A}$ is itself an algebra of continuous real-valued functions on $X$. The theorem is equivalent to the statement that $\bar{A} = C(X)$. This will follow from Proposition 30 if we can show that $\bar{A}$ is a lattice. Let $f \in \bar{A}$ and $\|f\| \leq 1$. Then given $\epsilon > 0$, $\||f| - P(f)\| < \epsilon$, where $P$ is the polynomial given in Lemma 33. Since $\bar{A}$ is an algebra containing the constants, $P(f) \in \bar{A}$, and since $\bar{A}$ is a closed subset of $C(X)$, we have $|f| \in A$. If now $f$ is any function in $A$, then $f/\|f\|$ has norm 1, and so $|f|/\|f\|$ and hence also $|f|$ belong to $\bar{A}$. Thus $\bar{A}$ contains the absolute value of each function which is in $\bar{A}$. But

$$f \vee g = \tfrac{1}{2}(f + g) + \tfrac{1}{2}|f - g|$$

and

$$f \wedge g = \tfrac{1}{2}(f + g) - \tfrac{1}{2}|f - g|.$$

Thus $\bar{A}$ is a lattice and must be $C(X)$ by Proposition 30. ▮

**35. Corollary:** *Every continuous function on a closed bounded set $X$ in $\mathbf{R}^n$ can be uniformly approximated on $X$ by a polynomial* (*in the coordinates*).

***Proof:*** The set of all polynomials in the coordinate functions forms an algebra containing the constants. It separates points, since given two distinct points in $\mathbf{R}^n$, one of the coordinate functions takes different values on these points. Hence Theorem 34 applies. ▮

## Problems

**42.** Let $f$ be a continuous periodic real-valued function on $\mathbf{R}$ with period $2\pi$; that is, $f(x + 2\pi) = f(x)$. Show that, given $\epsilon > 0$, there is a finite Fourier series $\varphi$, given by $\varphi(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$, such that $|\varphi(x) - f(x)| < \epsilon$ for all $x$. [Hint: Note that periodic functions are really