A set is "relatively compact" or "precompact" if its closure is compact.

For example if $\Sigma = \{ x \in l^2(\mathbb{R}, \mathbb{N}) \mid \sum x_i^2 \leq 1 \}$

then Σ is not totally bounded $\Rightarrow$ Σ is not compact in $l^2(\mathbb{R}, \mathbb{N})$.

On the other hand if $\Sigma = \{ x \in l^2(\mathbb{R}, \mathbb{N}) \mid \sum_1^\infty k^2 |x_k|^2 \leq 1 \}$

Then Σ is relatively compact in $l^2(\mathbb{R}, \mathbb{N})$.

From last time, we have a theorem that tells us when a space is relatively compact:

thm: $M \subset X$ where (X, ρ) is complete, then M is relatively compact $\iff$ M is totally bounded.

The problem is: it can be hard to show a set is totally bounded

When it comes to subsets of

$L^\infty([a,b])$ = continuous on $[a,b]$ with

metric $\rho(f,g) = \max_{x \in [a,b]} |f(x) - g(x)|$

the Arzela-Ascoli theorem is very helpful. It gives us conditions that are easier to check than total boundedness.

<u>defn</u>: A family Φ of functions ϕ defined on $[a,b] \subset \mathbb{R}$ is <u>uniformly bounded</u> if $\exists K > 0$ such that $|\phi(x)| \leq K \quad \forall x \in [a,b], \ \forall \phi \in \Phi$

<u>defn</u>: A family Φ of functions ϕ defined on $[a,b]$ is <u>equicontinuous</u> if given $\varepsilon > 0 \ \exists \ \delta > 0$ $\ni$ $|x-y| < \delta \Rightarrow |\phi(x) - \phi(y)| < \varepsilon$ for all $x, y \in [a,b]$, $\forall \phi \in \Phi$

(a family Φ is equicontinuous if each ϕ is uniformly continuous and if δ_ϕ doesn't depend on ϕ.)

<u>Arzela-Ascoli</u> (K+F) Let $\Phi \subset L^\infty([a,b])$. Then Φ is relatively compact in $L^\infty([a,b])$ if and only if Φ is uniformly bounded and equicontinuous.

proof:

($\Rightarrow$) Assume Φ is relatively compact. Show Φ is uniformly bounded and equicontinuous.

Since Φ is relatively compact, Φ is totally bounded. $\Rightarrow$ given $\varepsilon > 0$ $\exists$ finite $\varepsilon/3$-net $\{\phi_1 \dots \phi_n\}$ in Φ. (Note! We're guaranteed finite ε-nets in $L^\infty([a,b])$ --- have to make sure we can choose the finite ε-net in Φ itself.) We know each ϕ_i is cts on closed & bounded set

$\Rightarrow \exists K \ni \quad |\phi_i(x)| \le K \quad x \in [a,b], \; i = 1 \dots n$

Since $\{\phi_1 \dots \phi_n\}$ is an $\varepsilon/3$ net, given $\phi \in \Phi$

$\exists \phi_{i_0} \ni \quad \max\limits_x |\phi(x) - \phi_i(x)| \le \varepsilon/3$

$\Rightarrow \max\limits_x |\phi(x)| \le K + \varepsilon/3$ by triangle ineq.

$\Rightarrow \Phi$ is uniformly bounded. Now we show equicontinuity. Since a cts function on a compact set is uniformly cts, given $\varepsilon > 0$ $\exists \delta_i$ such that $|x-y| < \delta_i \Rightarrow |\phi_i(x) - \phi_i(y)| < \varepsilon/3$

Let $\delta = \min\{\delta_1, \dots \delta_n\}$ Now, if $|x-y| < \varepsilon/3$

$$|\phi(x) - \phi(y)| \le |\phi(x) - \phi_{i_0}(x)| + |\phi_{i_0}(x) - \phi_{i_0}(y)| + |\phi_{i_0}(y) - \phi(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

this proves Φ is equicontinuous, as desired.

($\Leftarrow$) Now assume Φ is uniformly bounded and equicontinuous. We'll construct a finite ε-net for each $\varepsilon > 0$. This proves Φ is totally bounded And then we'll be done. Given $\varepsilon > 0$, $\exists \delta > 0$ so that

$|x-y| < \delta \Rightarrow |\phi(x) - \phi(y)| < \varepsilon/5 \quad \forall \phi \in \Phi$.

Divide $[a,b]$ into n intervals

$$a = x_0 < x_1 < \dots < x_n = b$$

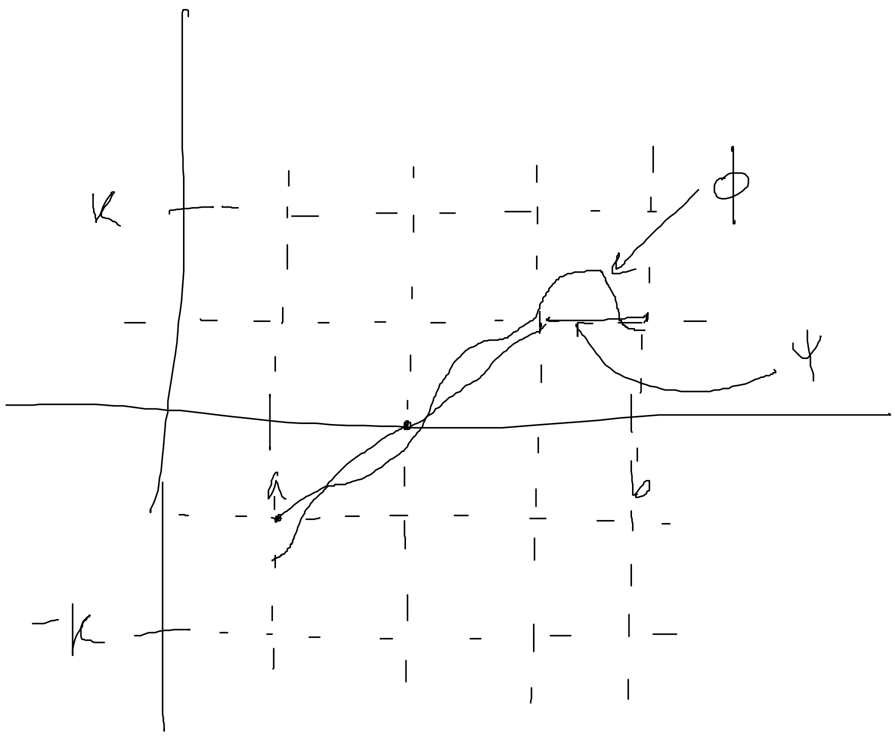
such that $|x_{i+1} - x_i| < \delta \quad i = 0 \dots n-1$.

Let K be the uniform bound on Φ ($|\phi(x)| \le K$ on $[a,b]$). Divide the interval $[-K, K]$ up into subintervals of length $< \varepsilon/5$. using subdivision

$$-K = y_0 < y_1 < \dots < y_p < K$$

$\Rightarrow$ Have divided $[a,b] \times [-K,K]$ up into np cells of length $< \delta$ and height $< \varepsilon/5$

For each $\phi \in \Phi$, we'll construct a piecewise linear approximant $\psi \in L^\infty([a,b])$ using this division of $[a,b]$



at each point $(x_i, \phi(x_i))$ on the graph, we find $y_{j_i} \in \{y_0, \dots y_p\}$ so that $|\phi(x_i) - y_j| < \varepsilon/5$.
This determines the pairs

$$(x_0, y_{j_0}), (x_1, y_{j_1}), \dots (x_n, y_{j_n})$$

we linearly interpolate between these points, creating a continuous function $\psi \in L^\infty([a,b])$.
Since $|x_{i+1} - x_i| < \delta$ we know $|\phi(x_{i+1}) - \phi(x_i)| < \varepsilon/5$

$\Rightarrow$ our constructed ψ satisfies $|\psi(x) - \psi(x_i)| < 3\varepsilon/5$ $x \in [x_i, x_{i+1}]$ at each meshpoint x_i (this used that ψ is linear!). Now we claim $\|\phi - \psi\| < \varepsilon$

$$|\phi(x) - \psi(x)| \le |\phi(x) - \phi(x_k)| + |\phi(x_k) - \psi(x_k)| + |\psi(x_k) - \psi(x)|$$

$$< \varepsilon/5 + \varepsilon/5 + 3\varepsilon/5 = \varepsilon.$$

Okay... so we've constructed an ε-net for Φ. A different ψ for each ϕ? Is the ε-net finite? Yes -- there are only finitely many piecewise-linear ψ for the fixed lattice. //

Okay... that was a very constructive proof. Also, we used that $[a,b] \subset \mathbb{R}$ when we did the linear approximants. We could certainly generalize to $\mathbb{R}^n$ but what about other metric spaces?

Thm: Let Φ be an equicontinuous family of functions from a separable space (X,ρ) to (Y,ρ'). Let $\{\phi_n\}$ be a sequence in Φ such that for each $x \in X$, $[\phi_n(x)]$ is compact in Y. Then $\exists$ a subsequence of $\{\phi_n\}$ so that $\{\phi_{n_k}\}$ converges pointwise to a continuous function ϕ, and the convergence is uniform on each compact subset of X.

Corr: Let Φ be an equicontinuous family of real-valued functions on a separable space (X,ρ). Then each sequence $\{\phi_n\}$ of Φ which is bounded at each point of a dense subset of X has a subsequence $\{\phi_{n_k}\}$ that converges pointwise to a continuous function ϕ, with the convergence being uniform on each compact subset of X.

<u>lemma</u>: Let $\{\phi_n\}$ be a sequence of mappings of a countable set D into a metric space (Y, ρ') such that for each $x \in D$ the closure of the set $\{\phi_n(x)\}$ (i.e. $[\phi_n(x)]$) is compact. Then $\exists$ a subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$ that converges for each $x \in D$.

<u>proof</u>: $D = \{x_k\}$ Since $[\phi_n(x_1)]$ is compact, $\exists$ subsequence $\{\phi_{n_1}\} \ni \phi_{n_1}(x_1) \to$ some limit. (call the limiting value $\phi(x_1)$.) Now, we take a subsequence of $\{\phi_{n_1}\}$ so that $\phi_{n_2}(x_2)$ converges. (Since it's a subsequence of $\{\phi_{n_1}\}$ we'll know it also converges at x_1.) Continue in this way at each x_i. Now, take the diagonal sequence $\{\phi_{n_n}\}$. This sequence of functions converges at each point of D, by construction. //

<u>lemma</u>: Let $\{\phi_n\}$ be an equicontinuous sequence of mappings from (X, ρ) to a complete space (Y, ρ'). If the sequences $\{\phi_n(x)\}$ converge at each $x \in D$ (dense) then $\{\phi_n\}$ converges at each $x \in X$ and ϕ is cts.

proof: given $x \in X$ and $\varepsilon > 0$ $\exists$ $\delta > 0$ $\ni$ $\rho(x, y) < \delta \Rightarrow \rho'(\phi_n(x), \phi_n(y)) < \varepsilon/3$. Since D is dense, $\exists$ $\tilde{y} \in D$ with $\rho(x, \tilde{y}) < \delta$. And since $\phi_n(\tilde{y})$ converges, $\exists N \ni m, n \geq N$

$\Rightarrow \rho'(\phi_n(\tilde{y}), \phi_m(\tilde{y})) < \varepsilon/3$.

$$\Rightarrow \rho'(\phi_n(x), \phi_m(x)) \leq \rho'(\phi_n(x), \phi_n(\tilde{y})) + \rho'(\phi_n(\tilde{y}), \phi_m(\tilde{y})) + \rho'(\phi_m(\tilde{y}), \phi_m(x))$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

$\Rightarrow \{\phi_n(x)\}$ is Cauchy and has a limiting value, call it $\phi(x)$. In this way, we've constructed $\phi : (X, \rho) \to (X, \rho')$. Now we show ϕ is cts. Fix $\varepsilon > 0$. By the equicontinuity of Φ $\exists$ $\delta > 0$ $\rho(x, y) < \delta \Rightarrow \rho'(\phi_n(x), \phi_n(y)) < \varepsilon$

$\forall n$. Since ρ' is continuous, we have

$$\rho'(\phi(x), \phi(y)) = \lim_{n \to \infty} \rho'((\phi_n(x), \phi_n(y))) \leq \varepsilon.$$

$\Rightarrow \rho(x, y) \Rightarrow \rho(\phi(x) - \phi(y)) \leq \varepsilon$. and ϕ

is cts at x. //

<u>lemma:</u> Let (K,ρ) be compact and $\{\phi_n\}$ an equicontinuous sequence of functions, $\phi_n : (K,\rho) \to (Y,\rho')$, that converge at each point of K to a function ϕ. Then $\{\phi_n\}$ converges <u>uniformly</u> on K.

<u>proof:</u> Let $\varepsilon>0$. By equicontinuity, $\exists\, \delta>0$ $\ni \rho(x,y)<\delta \Rightarrow \rho'(\phi_n(x),\phi_n(y))<\varepsilon/3$ $\forall\, \phi_n$. Taking $n\to\infty$, $\rho'(\phi(x),\phi(y)) \le \varepsilon/3$. We've covered K with open sets $S(x,\delta)$. $\exists$ finite subcover centered at $x_1,\dots x_p$. Now, choose N so large that $\rho'(\phi_n(x_i),\phi(x_i))<\varepsilon/3$ for $n\ge N$ and $i\in\{1,\dots p\}$. ~~be~~ Then for any $y\in K$, $y\in S(x_{i_0},\delta)$ some i_0 and if $n\ge N$ we have

$$\rho'(\phi_n(y),\phi(y)) \le \rho'(\phi_n(y),\phi_n(x_{i_0})) + \rho'(\phi_n(x_{i_0}),\phi(x_{i_0})) + \rho'(\phi(x_{i_0}),\phi(y)) < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}.$$

this shows $\phi_n \to \phi$ uniformly on K. //