ε-nets, total boundedness, metric spaces.

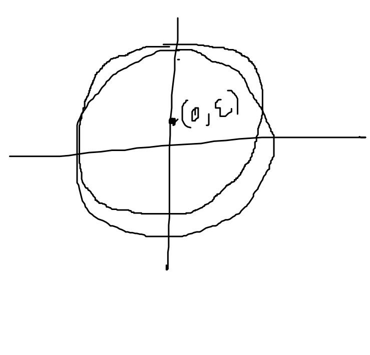
<u>Q</u>: How can you say a set is bounded when you don't have an origin to compare things to?

We'll use ε-nets to do this.

<u>defn</u>: Let (X, ρ) be a metric space and $\varepsilon > 0$. T $A \subset X$ is an <u>ε-net</u> $M \subset X$ if for every $x \in M$ $\exists\, a \in A \ni \rho(x,a) < \varepsilon$.

<u>Note</u>: If you have ε, then you need a metric space. One can define "nets" for general topological spaces, but not ε-nets.

<u>ex</u>: $X = \mathbb{R}^2$, M = circle of radius 1 centered at $(0,0)$, A = circle of radius $1-\varepsilon^2$ centered at $(0,\varepsilon)$ is a 2ε-net of M if $\varepsilon < 1$



<u>ex</u>: $X = \mathbb{R}$, $M = \mathbb{Z}$, $A = 2\mathbb{Z}$ then A is a $(1+\varepsilon)$-net of M

defn: Given metric space (X,ρ) and $M \subset X$.
If M has a finite ε-net for each $\varepsilon > 0$ then
M is totally bounded.

ex: in $\mathbb{R}^n$, total boundedness $\iff$ bounded.

ex: In $l^\infty(\mathbb{R}, \mathbb{N})$, the
sequences that satisfy $\sum_1^\infty x_n^2 = 1$
is bounded but isn't totally bounded
since $(e_n)_i = \begin{cases} 0 & i \neq n \\ 1 & i = n \end{cases}$

has $e_n \in \Sigma$ for each n but

$\rho(e_n, e_m) = \sqrt{2}$ if $n \neq m$. $\Rightarrow$ cannot
have an ε-net of Σ if $\varepsilon < \frac{\sqrt{2}}{2}$.

theorem: if (X,ρ) is countably compact
then it is totally bounded

proof: Assume not. Then $\exists\, \varepsilon_0 > 0 \ni (X,\rho)$
has no finite ε_0-net. Choose $a_1 \in X$.
then $\exists\, a_2 \in X \ni \rho(a_1, a_2) > \varepsilon_0$ (otherwise
$\{a_1\}$ would be an ε_0-net.)

And $\exists a_3 \ni \rho(a_1, a_3) > \varepsilon_0$ and $\rho(a_3, a_2) > \varepsilon_0$ since otherwise $\{a_1, a_2\}$ would be a finite ε_0-net for X.

Proceeding in this way given $\{a_1, .. a_n\}$ we choose $a_{n+1} \in X \ni \rho(a_k, a_{n+1}) > \varepsilon_0 \ \forall k \in \{1, ... n\}$. $\Rightarrow$ we have an infinite sequence w/ no limit point. $\times$, since (X, ρ) is countably compact. //

<u>Corollary</u>: if (X, ρ) is countably compact then $\exists$ a countable dense subset and a countable base. And is therefore compact.

<u>Note</u>: total boundedness says nothing about completeness!

thm: (X,ρ) is compact $\Leftrightarrow$ it is totally bounded and complete

proof:

($\Rightarrow$) Assume (X,ρ) compact. Then it's countably compact and hence totally bounded by previous theorem. If it's not complete then $\exists$ Cauchy sequence w/ no limit point. That's impossible because an infinite subset of a compact space must have a limit point.

($\Leftarrow$) Assume (X,ρ) is totally bounded and complete. Want to show it's compact. Let $\{x_n\}$ be an infinite sequence of distinct points in X. Since X is totally bounded, $\exists$ a finite 1-net. $\Rightarrow X \subset \bigcup_{i}^{n} S(a_i, 1)$. And one of these balls has ∞ly many points of the sequence in it. $\Rightarrow [S(a_1,1)]$ has a subsequence $\{x_n^{[1]}\}$. Now $[S(a_1,1)]$ is totally bounded $\Rightarrow [S(a_1,1)] \subset \bigcup_{i}^{m} S(a_k, 1/2)$ some $\{a_k\} \subset [S(a_1,1)]$

$\Rightarrow$ one of the balls has a ∞ subsequence $\{x_n^{[2]}\}$ in it

And we continue, finding

$a_3 \in S(a_2, 1/2)$ so that $S(a_3, 1/4)$ has

a further refinement of the sequence in it.

i.e. $\{x_n^{(k)}\} \subset S(a_k, \frac{1}{2^{k-1}})$.

Now the only issue is that the spheres aren't nested. Or closed. But if we multiply the radii by 2 they will be.

$$\ldots \subset [S(a_k, \tfrac{1}{2^{k-2}})] \subset [S(a_{k-1}, \tfrac{1}{2^{k-3}}] \subset \ldots$$

Then by the nested sphere theorem, we've found a Cauchy sequence. $\Rightarrow$ Since X is complete, $\exists$ a limit point. This proves that any infinite sequence has a limit point $\Rightarrow (X, \rho)$ is countably compact $\Rightarrow (X, \rho)$ is compact. //

defn: $M \subset (X, \tau)$ is relatively compact in (X, τ) if its closure is compact.

ex: $M = (-3, 1) \subset \mathbb{R}$ is rel compact

$M = (-3, \infty)$ is not

$M = [-4, \pi] \cap \mathbb{Q}$ is rel. compact in $\mathbb{R}$

theorem: If (X, ρ) is complete and $M \subset X$, then M is relatively compact $\Leftrightarrow$ M is totally bounded.

corollary: if ~~a~~ $\Sigma = \{ x \in l^{2}(\mathbb{R}, \mathbb{N}) \mid \sum_{1}^{\infty} x_n^2 = 1 \}$ then Σ is not relatively compact because Σ is not totally bounded! This makes sense because if Σ were relatively compact then the sequence $\{X_n\}$ where $(X_n)_i = \begin{cases} 1 & i = n \\ 0 & \text{otherwise} \end{cases}$ would have to have a limit point.

proof of theorem ($\Leftarrow$) let $[M]$ be the closure then $[M]$ is totally bounded and is complete since (X, ρ) is complete $\Rightarrow$ $[M]$ is compact.

($\Rightarrow$) if $[M]$ is compact then it's totally bounded $\Rightarrow$