$$\begin{array}{lll}
(L, \langle\, , \rangle) & \text{inner product space} & \\
\downarrow & \Downarrow & \\
(L, \|\ \|) & \text{normed vector space} & \\
\downarrow & \Downarrow & \\
(L, \rho) & \text{metric space} & \\
\downarrow & \Downarrow & \\
(L, \tau) & \text{topological space} &
\end{array}$$

fact (to be proved soon)

$\ell^p(\mathbb{R}, \mathbb{N})$ is not an inner product space if $p \neq 2$

$\ell^2(\mathbb{R}, \mathbb{N})$ is an inner product space

$$\langle x, y \rangle = \sum x_i y_i$$

$\ell^2(\mathbb{C}, \mathbb{N})$ has $\langle x, y \rangle = \sum x_i \overline{y_i}$

defn: given $x, y \in (L, \langle\, , \rangle)$

define the angle between x & y

by $\cos\theta = \dfrac{\langle x, y \rangle}{\|x\|\,\|y\|}$

where $\|x\| = \sqrt{\langle x, x \rangle}$, $\|y\| = \sqrt{\langle y, y \rangle}$

thm: if $\{X_\alpha\}$ are orthogonal then they are linearly indep.

proof: let

$$\alpha_1 X_1 + \dots + \alpha_n X_n = \vec{0} \quad \text{then}$$

$$\langle \alpha_1 X_1 + \dots + \alpha_n X_n, X_j \rangle = \langle \vec{0}, X_j \rangle = 0$$

$$\|$$

$$\alpha_j \langle X_j, X_j \rangle \Rightarrow \alpha_j = 0. \quad \text{since } \langle X_j, X_j \rangle \neq 0$$

Note: this works for any set of orthogonal vectors, even infinite ones.

An orthogonal set is an orthogonal basis if it's complete i.e. $\left[\begin{array}{c}\text{smallest subspace} \\ \text{containing } \{X_\alpha\}\end{array}\right] = L$

i.e. $X \in L$ $\varepsilon > 0$, then $\exists$ a linear combination of X_α $\ni$

$$\| X - \sum_i^n \alpha_i X_i \| < \varepsilon$$

if $\{X_\alpha\}$ is orthogonal and $\|X_\alpha\| = 1$ each α then we call $\{X_\alpha\}$ orthonormal ex: $L = \mathbb{R}^n$, $\langle x,y\rangle = \sum_i^n x_i y_i$

$$(e_k)_j = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

then $\{e_k\}_1^n$ is orthonormal basis

ex: $L = \ell^2(\mathbb{R}, \mathbb{N})$ $\langle x,y\rangle = \sum_1^\infty x_i y_i$

then $\{e_k\}_{k=1}^\infty$ is an orthonormal basis

ex: X = cts complex valued fxns on $[a,b]$ w/

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}\,dx$$

then $e_k(x) = e^{ik\frac{2\pi x}{b-a}}$

is orthonormal basis

When do we know we have an orthonormal basis? From Stone-Weierstrass, we know that the polynomials are dense in $L^\infty([a,b])$ (cts fxns w/ L^∞ norm.) The ones w/ rational coeffs are also dense $\Rightarrow$ $L^\infty([a,b])$ is <u>separable</u>. Furthermore, if $[b-a] < \infty$ then we can use this to show that the polys w/ rational coeffs are dense in $L^p([a,b])$ $1 \le p < \infty$.

thm: if $(L, \langle , \rangle)$ is separable and $\{x_\alpha\}$ is orthogonal then $\{x_\alpha\}$ is either finite or countable.

proof: WLOG, assume $\{x_\alpha\}$ is orthonormal. Since $\frac{x_\alpha}{\|x_\alpha\|}$ is orthonormal.

then $\|x_\alpha - x_\beta\| = \sqrt{\langle x_\alpha - x_\beta, x_\alpha - x_\beta \rangle}$

$$= \sqrt{\langle x_\alpha, x_\alpha\rangle + \cancel{2}\langle x_\alpha, x_\beta\rangle + \langle x_\beta, x_\alpha\rangle + \langle x_\beta, x_\beta\rangle}$$

$$= \sqrt{\langle x_\alpha, x_\alpha\rangle + \langle x_\beta, x_\beta\rangle}$$

$$= \sqrt{2}.$$

$\Rightarrow$ these vectors are $\sqrt{2}$ apart.

Consider $S(x_\alpha, \sqrt{2})$. these are pairwise disjoint and each sphere contains an elt of our countable dense set. $\Rightarrow$ only countably many spheres $\Rightarrow$ $\{x_\alpha\}$ is finite or countable

//

Fact: if $(L, <,>)$ is separable then
$\exists$ orthogonal basis. follows from following theorem + corollary.

theorem: Let $f_1, f_2, \ldots$ be any countable set of linearly indep elts in $(L, <,>)$.
then $\exists$ a set $\phi_1, \phi_2, \ldots$ so that

1) $\{\phi_i\}$ is orthonormal
2) $\phi_n = \sum_{i=1}^{n} a_{ni} f_i \qquad a_{nn} \neq 0$
3) $f_n = \sum_{1}^{n} b_{ni} \phi_i \qquad b_{nn} \neq 0$

and every f_i is uniquely determined up to a factor of ± 1.

proof: first construct $\{\phi_i\}$

$$\phi_1 = a_{11} f_1 \quad \text{where}$$

$$\langle \phi_1, \phi_1 \rangle = a_{11}^2 \langle f_1, f_1 \rangle = 1$$

$\Rightarrow a_{11}$ det'd up to a sign

$\Rightarrow \phi_1$ det'd uniquely up to a sign.

And 2), 3) are also true.
assume $\phi_1 .. \phi_{n-1}$ have been constructed
now construct ϕ_n, so that 1), 2), 3) hold

$$f_n = b_{n1}\phi_1 + b_{n2}\phi_2 + \ldots + b_{n,n-1}\phi_{n-1} + h_n$$

where $\langle h_n, \phi_k \rangle = 0$ if $k = 1 \cdots n-1$

why? we choose $b_{nk} = \langle f_n, \phi_k \rangle$ then

$$\langle f_n, \phi_j \rangle = \sum_{1}^{n-1} b_{nk} \langle \phi_k, \phi_j \rangle + \langle h_n, \phi_j \rangle$$

$$= b_{nj} \cdot 1 + \langle h_n, \phi_j \rangle$$

$$= \langle f_n, \phi_j \rangle + \langle h_n, \phi_j \rangle \Rightarrow \langle h_n, \phi_j \rangle = 0 \text{ if } j = 1 .. n-1$$

also $\langle h_n, h_n \rangle \neq 0$ since if $\langle h_n, h_n \rangle = 0$ then

f_n is a linear combination of $f_1 .. f_{n-1}$

(since $\phi_1 \in \text{span}\{f_1\}$, $\phi_2 \in \text{span}\{f_1, f_2\}$, .. $\phi_{n-1} \in \text{span}\{f_1 .. f_{n-1}\}$ and if $h_n = 0$ then $f_n \in \text{span}\{\phi_1 .. \phi_{n-1}\}$.)

since $h_n \neq 0$, define $\phi_n = \dfrac{h_n}{\|h_n\|}$

then we have $\phi_n \in \text{span}\{f_1, .. f_n\}$ and

$\phi_n = a_{n1} f_1 + .. + a_{nn} f_n$ where $a_{nn} \neq 0$. $a_{nn} = \dfrac{1}{\|h_n\|}$

and $\langle \phi_n, \phi_k \rangle = 0$ $k = 1 .. n-1$, $\langle \phi_n, \phi_n \rangle = 1$

and $f_n = b_{n1} \phi_1 + .. + b_{nn} \phi_n$ where $b_{nn} \neq 0$

so we've found $\{\phi_1 ... \phi_n\}$ that satisfy 1), 2), 3) using $\{\phi_1 .. \phi_{n-1}\}$ that satisfied 1), 2), 3). Done by induction! //

Corr: if $(L, <,>)$ is separable then $\exists$ countable ortho-normal basis

proof: Let $\{\psi_1, \ldots, \psi_n, \ldots\}$ be the countable dense set. Using theorem, $\psi_1, \psi_2, \ldots$ gives us $\phi_1, \phi_2, \ldots$ so that $\{\phi_i\}$ ~~are~~ is orthonormal and

$$\psi_n = \sum_1^n b_{ni} \phi_i$$

let $x \in L$, $\varepsilon > 0$. then $\exists$ n_0 $\ni$

$$\| x - \psi_{n_0} \| < \varepsilon,$$

$$\Rightarrow \quad \| x - \sum_1^{n_0} b_{ni} \phi_i \| < \varepsilon$$

$\Rightarrow$ $x \in$ smallest closed subspace that contains $\{\phi_i\}$

$\Rightarrow$ $\{\phi_i\}$ is an orthonormal basis. //