Continuous functions...

Recall the theorem on $\mathbb{R}^n$:

Let $A \subset \mathbb{R}^n$ be closed and bounded. Let $f: A \to \mathbb{R}$ be continuous. Then $f(A)$ is bounded above and below and $\exists\, x_0 \in A \ni f(x_0) = \text{l.u.b. } f(A)$ and $\exists\, x_1 \in A \ni f(x_1) = \text{g.l.b. } f(A)$

Is there an analogue for topological spaces?

Yes But first we define upper and lower semicontinuity

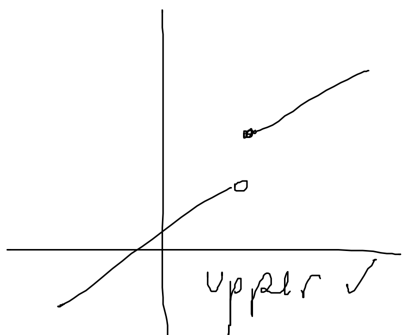
defn: $f: (X,\tau) \to \mathbb{R}$ is upper semicontinuous if for each $\alpha \in \mathbb{R}$ $\{x \mid f(x) < \alpha\}$ is open

defn: $f: (X,\tau) \to \mathbb{R}$ is lower semicontinuous if for each $\alpha \in \mathbb{R}$ $\{x \mid f(x) > \alpha\}$ is open.

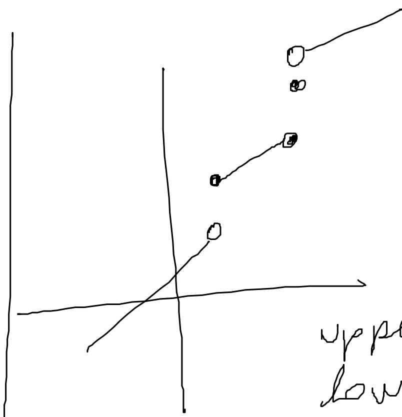
Note: If f and $-f$ are upper semicontinuous then f is continuous.

examples of semicontinuous functions?

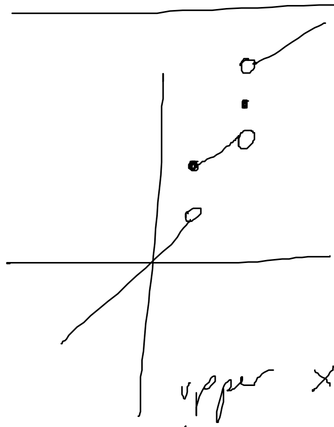
Consider $f: \mathbb{R} \to \mathbb{R}$ with the following graphs:



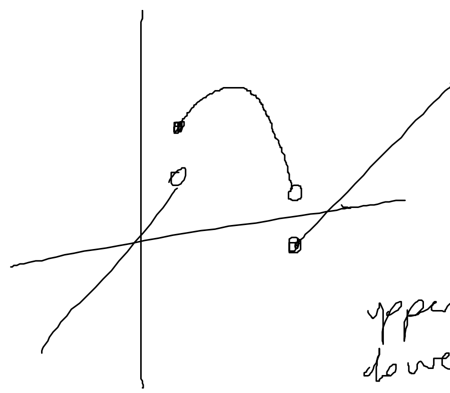
upper ✓
lower ✗



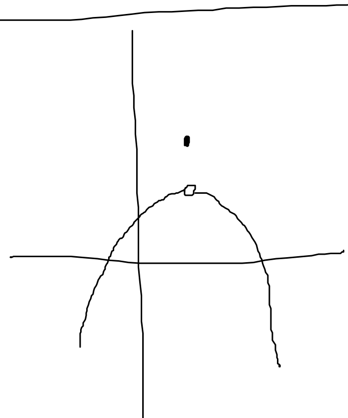
upper ✗
lower ✗



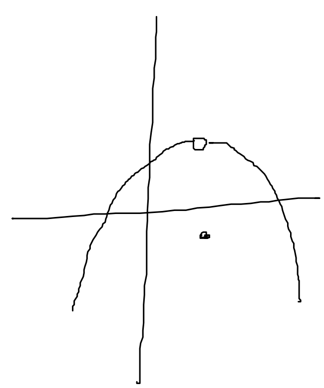
upper ✗
lower ✗



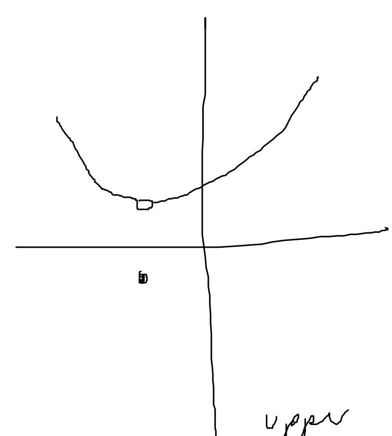
upper ✗
lower ✗



upper ✓
lower ✗



upper ✗
lower ✓



upper ✗
lower ✓

<u>theorem</u>: Let f be an upper semicontinuous real-valued function on a compact top. space (X, τ). Then f is bounded from above and assumes its maximum

<u>corollary</u>: let f be a lower semicontinuous real valued function on a compact top. space (X, τ). Then f is bounded from below and assumes its minimum.

<u>corollary</u>: let f be a continuous real-valued function on a compact topological space (X, τ). Then f is bounded above and below and assumes its maximum and minimum.

<u>proof of theorem</u>: Let $U_n = \{x \mid f(x) < n\}$.

They are open since f is upper semicont.

Then $\{U_n\}$ is an open cover of X.

$\Rightarrow \exists$ a finite subcovering $\{U_1, .., U_N\}$.

$\Rightarrow X \subset U_N \Rightarrow f(x) < N \quad \forall x \in X \Rightarrow f$ is bounded above

Now define

$$\beta = \sup\{ f(x) \mid x \in X \}.$$

then $F_n = \{ x \mid f(x) \geq \beta - 1/n \}$
is a collection of closed sets with the finite intersection property.

$$\Rightarrow \bigcap_1^\infty F_n \neq \phi.$$

Let $x_0 \in \bigcap_1^\infty F_n$. Then $f(x_0) = \beta$ and f has achieved its maximum. //

<u>theorem:</u> Let $\{f_n\}$ be a sequence of upper semicontinuous real-valued functions on a compact space (X, τ). Assume that for each $x \in X$, the sequence $\{f_n(x)\}$ decreases monotonically to 0. Then $\{f_n\}$ converges to 0 uniformly.

<u>proof:</u> Choose $\varepsilon > 0$. Let $U_n = \{ x \mid f_n(x) < \varepsilon \}$.
f_n is upper semicont $\Rightarrow$ U_n is open for each n. Since $f_n(x) \to 0$ for each x, we have $X \subset \bigcup_1^\infty U_n \Rightarrow X \subset \bigcup_1^N U_n$ for some subcollect.
$\Rightarrow X = U_N$ because $f_n(x)$ decr. monotonically.
$\Rightarrow f_N(x) < \varepsilon \; \forall x$. If $n \geq N$ then $0 \leq f_n(x) \leq f_N(x) < \varepsilon$

and this shows $\{f_n\} \to 0$ uniformly in x. //

<u>defn:</u> (X,τ) is <u>countably compact</u> if every countable open cover has a finite subcover

<u>thm!</u> (X,τ) is countably compact if and only if every countable collection of closed centered subsets has nonempty intersection.

<u>theorem:</u> (X,τ) compact $\Rightarrow$ (X,τ) countably compact

<u>theorem:</u> (X,τ) second countable and countably compact $\Rightarrow$ (X,τ) compact.

<u>Note:</u> the two theorems about upper semicontinuous functions didn't really use (X,τ) compact. They used (X,τ) countably compact, which is a weaker property.