Recall the separating hyperplane theorem from last time -

if $M \cap N = \phi$, M, N convex, and $I(M)$ or $I(N) \neq \phi$. then $\exists$ linear functional f and value c so that

$$f(m) \leq c \quad \forall m \in M, \quad f(n) \geq c \quad \forall n \in N$$

Q: When can we make these into strict inequalities?

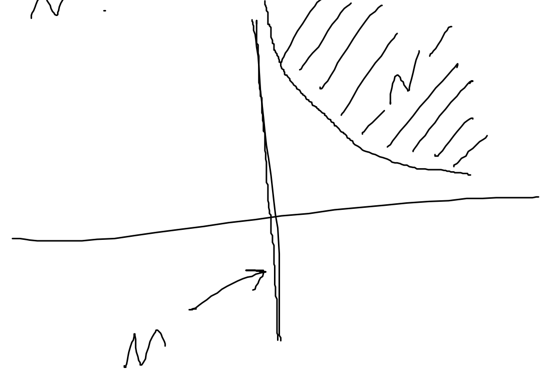
A: if both M and N are open

A: If M is compact and N is closed

ex: take $M =$ y-axis

$N = \{ y \geq 1/x \mid x > 0 \}$

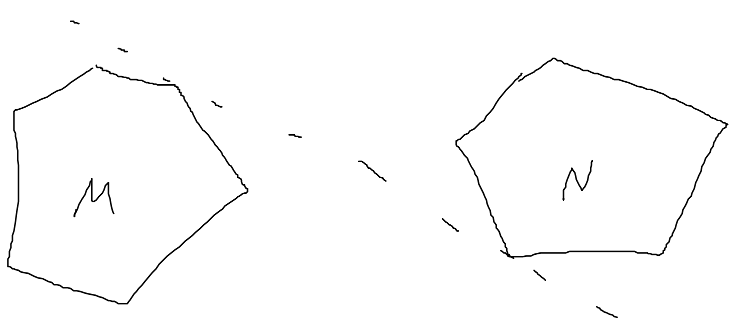
then $M \cap N = \phi$, M, N closed, M, N convex, $I(N) \neq \phi$. But you cannot strictly separate M from N:



Another concept is a <u>supporting hyperplane</u> i.e.

$H = \{x \mid f(x) = c\}$ is a supporting hyperplane to a convex set M if $f(x) \le c \;\forall\, x \in M$ and if $[M] \cap H \neq \emptyset$.

i.e. H touches $[M]$



H supports both M and N

<u>Q</u>: When is there a supporting hyperplane?

<u>Q</u>: When can you separate two convex sets w/ two supporting hyperplanes? With one supporting hyperplane?

<u>Note</u>: All of the above referred to <u>open</u> and <u>closed</u> and <u>compact</u>. We need a topology!

Given a vector space L, a <u>norm</u> on L is a convex functional $\|\cdot\| : L \to [0,\infty)$ that satisfies

1) $p(x) < \infty \ \forall x$, $p(x+y) \leq p(x) + p(y)$
2) $p(x) = 0 \Leftrightarrow x = \vec{0}$
3) $p(\alpha x) = |\alpha| p(x) \quad \forall x \in L, \forall \alpha$

We denote p by $\|\cdot\|$. Condition 2 is what distinguishes a norm from a convex functional. Also 3, since (3) was only assumed for $\alpha > 0$ for convex functionals.

$(L, \|\cdot\|)$ is a normed vector space. We've already seen a lot of these

$L = C([a,b])$ with $\|f\|_2 = \sqrt{\int_a^b |f(x)|^2 dx}$

$L = C([a,b])$ with $\|f\|_p = \sqrt[p]{\int_a^b |f(x)|^p dx}$

$L =$ polynomials on $[a,b]$ with $\|f\|_\infty = \max\limits_{x \in [a,b]} |f(x)|$

given $(L, \|\cdot\|)$ you can use $\|\cdot\|$ to define a metric by $\rho(x,y) = \|x-y\|$. Once we have a metric, we can say whether or not (L, ρ) is complete

L = polynomials, $\|f\| = \rho_\infty(f, \vec{0})$
is not complete, since [polynomials] = cont. funcs.

If $(L, \|\cdot\|)$ is complete (w.r.t. the induced metric ρ) then $(L, \|\cdot\|)$ is a Banach space.

We do have to be a little careful since $(L, \|\cdot\|)$ can play some games on us.

for example
span$\{x_1, x_2, \ldots\}$ might not be closed!

Before, when we didn't have a norm, we defined a subspace to be the smallest subspace that contained $x_1, \ldots$

Now, when we talk about a subspace, we mean the smallest closed subspace containing $x_1, \ldots$

So if $\{x_\alpha\}$ = polynomials then this is a subspace of the vector space of continuous functions. But! If we give the continuous functions the norm $\|f\| = \sup_{x \in [a,b]} |f(x)|$ then the subspace that contains $\{x_\alpha\}$ is $C[a,b]$

In a metric space, we defined
A bounded if
$$d(A) = \sup_{x,y \in A} \rho(x,y) < \infty, \quad (*)$$

In $(L, \|\cdot\|)$ we say A is bounded if
$$\sup_{x \in A} \|x\| < \infty. \quad (**)$$

If we start with $(L, \|\cdot\|)$ and let $\rho(x,y) = \|x-y\|$
then A is bounded in sense $(*)$ $\Leftrightarrow$ A is bounded
in sense $(**)$. And if $B_n \subset (L, \|\cdot\|)$ is
a nested family of closed balls, then $\bigcap B_n \neq \emptyset$.

But! $\exists$ nested familys of bounded closed
sets so that $B_1 \supset B_2 \supset \dots$ and $B_n = \emptyset$.
trouble... closed = contains limit points.
open = trickier.

<u>ex</u>: Let $L = \{$bounded infinite sequences$\}$ with
$$\|x\| = \sup_{i \in \mathbb{N}} |x_i|$$

$$B_1 = \{x \mid x_1 = 0, \|x\|_\infty = 1\}$$
$$B_2 = \{x \mid x_1 = x_2 = 0, \|x\|_\infty = 1\}$$

etc.

So we have

$$B_1 \supset B_2 \supset \cdots \supset B_n$$

where each B_i is closed and bounded (convex even!)

but $\bigcap_n B_n = \phi$.

What about open sets?

We had the concept of interior before.

Q: if $x \in I(M)$, can you find $\varepsilon > 0$ so that $S(x, \varepsilon) \subset M$?

A: Intuitively, you should risk trouble in infinite dimensions.

$x \in I(M) \Rightarrow$ given $y \in L$ $\exists\, \varepsilon_y > 0$ so that $x + ty \in M$ $|t| < \varepsilon_y$.

What if $\varepsilon_y \downarrow 0$?

ex: Let $L = \{$bounded sequences w/ only finitely many nonzero components$\}$

with $\|x\| = \sqrt{\sum |x_i|^2}$

let $M = \{x \in L \mid \|x\| \leq 1\}$.

claim: $\vec{0} \in I(M)$ but you cannot put an ε-sphere around $\vec{0}$ that stays in M.

why? Take

$y \in L$. then let

$N = \#$ of nonzero components of y

and $|y_i| \le C \quad \forall i$.

then $\|\vec{0} + ty\|$

$$= |t| \sqrt{\sum |y_i|^2}$$

$$< |t| C \sqrt{N} \le 1 \text{ if } |t| < \frac{1}{C\sqrt{N}}$$

$\Rightarrow$ given y, $\vec{0} + ty \in M$ if t small enough.

But clearly we have $\varepsilon_y \downarrow 0$ as $N \uparrow \infty$ so there's no single ε that will work.

$\Rightarrow$ cannot put an ε-ball around $\vec{0}$ that will stay in M. !!

Given $(L, \|\cdot\|)$ we can use $\|\cdot\|$ to generate a metric ρ.

Given (X, ρ) we can't neccessarily put a norm on the space. (Need X to be a vector space, for one thing!)

Stronger than the normed vector spaces are the <u>inner product spaces</u>

i.e. L with $\langle \, , \, \rangle$ where

$\langle \, , \, \rangle : L \times L \to \mathbb{R}$ satisfies

1) $\langle x, x \rangle \geq 0 \quad \forall x, \quad \langle x, x \rangle = 0 \Leftrightarrow \vec{x} = \vec{0}$
2) $\langle x, y \rangle = \langle y, x \rangle$
3) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
4) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$

<u>Claim:</u>

$|\langle x, y \rangle| \leq \|x\| \, \|y\|$ where

$\|x\| := \sqrt{\langle x, x \rangle}, \quad \|y\| = \sqrt{\langle y, y \rangle}$

<u>proof:</u>

let $p(\lambda) = \langle \lambda x + y, \lambda x + y \rangle$

$$= \lambda^2 \langle x, x \rangle + 2\lambda \langle x, y \rangle + \langle y, y \rangle$$

$$= \lambda^2 \|x\|^2 + 2\lambda \langle x, y \rangle + \|y\|^2 \geq 0 \quad \forall \lambda$$

$p(\lambda)$ minimized at $\lambda = -\langle x, y \rangle / \|x\|^2$

and $P_{min} = \|y\|^2 - (\langle x, y \rangle)^2 / \|x\|^2 \geq 0$

$$\Rightarrow (\langle x, y \rangle)^2 \leq \|x\|^2 \|y\|^2. \quad //$$