The Hahn Banach theorem

Given real vector space L and subspace L_0 if we have a linear functional f_0 defined on L_0, we want to know if we can extend f_0 from L_0 to L in a reasonable manner.

Recall from topological spaces that if (X,τ) was T_4 and $A \subset X$ was closed then a continuous real-valued function from A to $[a,b]$ could be extended to one from X to $[a,b]$

$$f: A \to [a,b]$$

extends to

$$F: X \to [a,b]$$

so that $F|_A = f$

<u>Note:</u> Not only is F an extension but we had control on the range of F.

Here, we don't have concepts like closed or T_4. But we do have an extension theorem.

<u>theorem</u>: Let p be a finite convex functional defined on L and let L_0 be a subspace of L. Suppose f_0 is a linear functional on L_0 that satisfies

$$f_0(x) \le p(x) \quad \forall x \in L_0 \quad \circledast$$

Then f_0 can be extend to a linear functional on L that satisfies $\circledast$ for all $x \in L$.

<u>proof</u>: Assume $L_0 \neq L$ since otherwise we're done. Take $z \in L$ so that $z \notin L_0$ and let $\tilde{L}$ be the subspace generated by L_0 and z. i.e. $\tilde{L} = \{ x + tz \mid x \in L_0, t \in \mathbb{R} \}$. We want to define

$$\tilde{f} : \tilde{L} \to \mathbb{R}$$

so that $\tilde{f}$ extends L_0 and $\tilde{f}(x) \le p(x) \ \forall x \in \tilde{L}$

Let's look at the condition of linearity:

$$\tilde{f}(x + tz) = \tilde{f}(x) + t\tilde{f}(z) = f_0(x) + t\tilde{f}(z).$$

We need to choose a value c for $\tilde{f}(z)$ so that if we define

$$\tilde{f}(x + tz) = f_0(x) + tc \quad \text{then we have}$$

a linear extension that respects

$$\tilde{f}(x) \le p(x) \quad \forall x \in \tilde{L}.$$

We want

$$\tilde{f}(x+tz) = f_0(x) + tc \le p(x+tz) \quad \forall x \in L_0 \quad \forall t.$$

Case 1: $t > 0$

$$f_0\left(\frac{x}{t}\right) + c \le p\left(\frac{x}{t} + z\right) \quad \forall x \in L_0 \qquad (*)$$

Case 2: $t < 0$

$$f_0\left(\frac{x}{t}\right) + c \ge -p\left(-\frac{x}{t} - z\right) \quad \forall x \in L \qquad (**)$$

since $x \in L_0 \Rightarrow \frac{x}{t} \in L_0$ we see that if

$$c \le \inf_{\tilde{x} \in L_0} p(\tilde{x} + z) - f_0(\tilde{x})$$

then $(*)$ will certainly be true. On the other hand, if

$$c \ge \sup_{\tilde{x} \in L_0} -p(-\tilde{x} - z) - f_0(\tilde{x})$$

then $(**)$ will certainly be true.

So if we can show that

$$\sup_{\tilde{x}\in L_0} -p(-\tilde{x}-z)-f_0(\tilde{x}) \le \inf_{\tilde{x}\in L_0} p(\tilde{x}+z)-f_0(\tilde{x})$$

then we can choose some c between the infimum and the supremum. (Note: if the supremum < infimum then we don't have a unique choice for $c \Rightarrow$ extension not uniquely determined.)

Let y'' and $y' \in L_0$. Then

$$\begin{aligned} f_0(y''-y') = f_0(y'')-f_0(y') &\le p(y''-y') \\ &= p((y''+z)-(y'+z)) \\ &\le p(y''+z)+p(-y'-z) \end{aligned}$$

$\Rightarrow \; -p(-y'-z)-f_0(y') \le p(y''+z)-f_0(y'')$
for all $y', y'' \in L_0$.

$\Rightarrow$ supremum $\le$ infimum.

This shows that given L_0 and $z \notin L_0$, we can extend f_0 to $\text{span}\{L_0, z\}$ in a way that respects $f(x) \le p(x)$.

If L is spanned by a countable basis then we've done by induction. What if L isn't spanned by a countable basis? Transfinite induction!

Let $\mathcal{F}$ be the set of all possible extensions of f_0 that satisfy the majorization condition. It's a partially ordered set and each linearly ordered set $\mathcal{F}_0 \subset \mathcal{F}$ has an upper bound. By Zorn's lemma, $\mathcal{F}$ has a maximal element. And this maximal element must be defined on all of L since if its domain were smaller than L, then we could extend f contradicting that f is maximal //

There's also a Hahn-Banach theorem for complex linear functionals. (See the book.)

An application of H-B is to separate objects. (To separate a point x_0 from a subspace $L_0 \not\ni x_0$. Or to separate two convex subsets of L.)

What does it mean to separate things when you're in a vector space?

You can separate $M, N \subset L$ if you can find a linear functional f on L and a value α so that

$$f(x) \geq \alpha \quad \forall\, x \in M$$
$$f(x) \leq \alpha \quad \forall\, x \in N$$

note: f separates M and N

$\iff$ f separates $\{0\}$ and $M-N = \{x-y \mid x \in M, y \in N\}$

note: f separates M and N

$\iff$ f separates $M-x_0$ and $N-x_0$ for every $x_0 \in L$.

thm: Let M and N be two disjoint convex sets in a real vector space L where at least one of M and N is a convex body. Then $\exists$ a nontrivial linear functional on L that separates M and N

proof: Assume $\vec{0} \in I(M)$. (otherwise, take $x_0 \in I(M)$ and replace M by $M-x_0$ and N by $N-x_0$.) Take $y_0 \in N$. Then $-y_0 \in I(M-N)$. This is since given $z \in L$ $\exists\, \varepsilon_z > 0$ so that $\vec{0} + tz \in M \quad \forall\, |t| < \varepsilon_z$.

$\Rightarrow tz - y_0 \in M-N \quad \forall\, |t| < \varepsilon_z \Rightarrow -y_0 \in I(M-N)$

Also, $\vec{0} \in I(M-N+y_0)$ since

$\vec{0} \in I(M)$ and $M-N+y_0 \subset M$.

We know M and N are disjoint

$\Rightarrow 0 \notin M-N$ and $y_0 \notin M-N+y_0$

Let p be the Minkowski functional for the set $M-N+y_0$. Then $p(y_0) \geq 1$ since $y_0 \notin M-N+y_0$. We introduce our linear functional f_0

$$f_0(\alpha y_0) = \alpha p(y_0)$$

defined on $L_0 = \text{span}\{y_0\}$. Then $f_0(\alpha y_0) \leq p(\alpha y_0)$ for all $\alpha \Rightarrow f_0 \leq p$ on L_0. Using the Hahn-Banach theorem, $\exists\, f: L \to \mathbb{R}$ so that $f(y) \leq p(y)$. $\Rightarrow f(y) \leq 1$ if $y \in M-N+y_0$ and $f(y_0) \geq 1$. $\Rightarrow f$ separates $M-N+y_0$ from $\{y_0\}$. $\Rightarrow f$ separates $M-N$ and $\{0\}$.

$\Rightarrow f$ separates M and N. //