Convex Sets & Functionals

Given $x, y \in L$ and $\alpha, \beta \geq 0 \ni \alpha + \beta = 1$

then $\{\alpha x + \beta y \mid \alpha, \beta \geq 0, \alpha + \beta = 1\}$

is the line segment in L connecting x to y.

If $\alpha \neq 0$ and $\alpha \neq 1$ then it's an open segment

Given a set $M \subset L$, its interior $I(M)$ is defined by:

$x \in I(M)$ if given $y \in L$ $\exists\, \varepsilon_y > 0$ so that $x + \varepsilon y \in M$ $\forall\, |\varepsilon| < \varepsilon_y$.

This is the best we can do w/o a metric or anything like that. If L is finite dimensional then we can take one ε for all y.

defn: $M \subset L$ is convex if for any $x, y \in M$ the line segment connecting x and y is also in M.

defn: A convex set is also a convex body if $I(M) \neq \emptyset$

ex: $[a, b]$ is a convex body in $\mathbb{R}$ but not in $\mathbb{R}^2$

convex bodies are intuitive in $\mathbb{R}^n$. They're less intuitive in infinite dimensions. First of all, whether or not something's a convex body depends on the ambient space L.

<u>For example</u>. let $L = l^2(\mathbb{C}, \mathbb{Z})$, fix $\rho > 0, K > 0$ and define

$$\Sigma = \{ x \in l^2(\mathbb{C}, \mathbb{Z}) \mid |X_n| \leq K e^{-\rho|n|} \}$$

then Σ is convex, but is not a convex body because you can take $y \in l^2(\mathbb{C}, \mathbb{Z})$ so that $|y_n| \downarrow 0$ more slowly than $Ke^{-\rho|n|}$. (algebraically, for example.) Then no matter how small you take t, you won't have $x + ty \in \Sigma$.

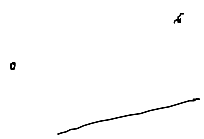
<u>thm:</u> if M is convex then $I(M)$ is convex

<u>thm:</u> if M_α is convex for each α then $\bigcap_\alpha M_\alpha$ is convex.

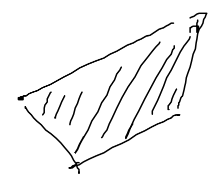
Given any set $A \subset L$, there is a smallest convex set that contains A. (i.e. the intersection of all convex sets that contain A). This set is called the <u>convex hull</u> of A

e.g

A :



then hull of A



$p : L \to \mathbb{R} + \{\infty\}$ is <u>convex</u> if   (L is a vector space over $\mathbb{R}$)

1) $p(x) \geq 0 \quad \forall x \in L$

2) $p(\alpha x) = \alpha p(x) \quad \forall x \in L, \ \forall \alpha \geq 0$

3) $p(x+y) \leq p(x) + p(y) \quad \forall x, y$

<u>ex:</u> if $L = L^2([a,b])$ (continuous real-valued functions w/ L^2 metric)

then $p(f) = \sqrt{\int_a^b f(x)^2 \, dx}$

is a convex functional.

<u>theorem:</u> If p is a convex functional on a vector space L and $k>0$ then

$$E = \{x \mid p(x) \le k\}$$

is convex. Further, if p is a finite convex functional then E is a convex body and $I(E) = \{x \mid p(x) < k\}$.

<u>proof:</u> $x, y \in E$ $\alpha, \beta \ge 0$ so that $\alpha + \beta = 1$

then $p(\alpha x + \beta y) \le p(\alpha x) + p(\beta y)$

$$= \alpha p(x) + \beta p(y) \le k$$

$\Rightarrow$ $\alpha x + \beta y \in E \Rightarrow E$ is convex. Now assume p is a finite functional.

Let $x \in \{x \mid p(x) < k\}$. Fix $y \in L$. I want to show $\exists\, \varepsilon_y > 0$ so that $|t| < \varepsilon_y \Rightarrow$

$$x + ty \in \{x \mid p(x) < k\}$$

This would show $\{x \mid p(x) < k\} \subset I(E)$.

Rather than allowing $|t| < \varepsilon_y$, we'll look at $0 \le t < \varepsilon_y$ and $x \pm ty$.

$$p(x \pm ty) \le p(x) + t p(\pm y).$$

If $p(y) = 0$ then take any t. If $p(-y) = 0$ then take any t. If either $p(y)$ or $p(-y) \ne 0$ then we have to restrict the range of t.

let $t < \dfrac{k - p(x)}{\max\{p(y), p(-y)\}} =: \varepsilon_y$

then if $0 \le t < \varepsilon_y$ we'll have

$p(x \pm ty) < k \Rightarrow x \pm ty \in \{x \mid p(x) < k\}$

$\Rightarrow \{x \mid p(x) < k\} \subset I(E)$.

Further, if $x \in \{x \mid p(x) = k\}$ then $x \notin I(E)$.

Since we can test in the direction x. Need $\varepsilon_x > 0 \ni x + tx \in \{x \mid p(x) \le k\} \ \forall\ |t| < \varepsilon_x$

but $p(x+tx) = (1+t)p(x) \nleq k$ if $t > 0$.

$\Rightarrow I(E) = \{x \mid p(x) < k\}$. //

Now go in the opposite direction. Suppose E is a convex body whose interior contains $\vec{0}$. Use E to define a functional:

$$p_E(x) = \inf\{r \mid \tfrac{x}{r} \in E,\ r > 0\}$$

p_E is the <u>Minkowski functional</u> of the convex body E.

<u>theorem</u>: if $E \subset L$ is a convex body such that $\vec{0} \in I(E)$ then p_E is a finite, convex functional.

<u>proof</u>: Fix $x \in L$. Since $\vec{0} \in I(E)$, $\exists$ $\varepsilon_x > 0$ so that $|t| < \varepsilon_x \Rightarrow \vec{0} + t\vec{x} \in E$.

$\Rightarrow$ for r sufficiently large, $\frac{x}{r} \in E$.

$\Rightarrow \inf\{r \mid \frac{x}{r} \in E, r > 0\}$ exists and is finite for each $x \in L$. This shows finiteness.

Now $p(\vec{0}) = 0$ is immediate since $\vec{0} \in E$.

and if $\alpha > 0$ then

$$p_E(\alpha x) = \inf\{r \mid \frac{\alpha x}{r} \in E, r > 0\} \qquad r = \alpha r'$$

$$= \inf\{\alpha r' \mid \frac{\alpha x}{\alpha r'} \in E, r' > 0\}$$

$$= \inf\{\alpha r' \mid \frac{x}{r'} \in E, r' > 0\}$$

$$= \alpha \inf\{r' \mid \frac{x}{r'} \in E, r' > 0\}$$

$$= \alpha\, p_E(x) \checkmark$$

Now show $p_E(x+y) \le p_E(x) + p_E(y)$ $\forall x, y \in L$.

Fix $\varepsilon > 0$. Choose $r_1 > 0 \ni$

$$p_E(x) < r_1 < p_E(x) + \varepsilon$$

choose $r_2 > 0 \ni$

$$p_E(y) < r_2 < p_E(y) + \varepsilon.$$

$\Rightarrow \frac{x}{r_1}$ and $\frac{y}{r_2} \in E$ by defn. Since E is convex, the segment connecting $\frac{x}{r_1}$ to $\frac{y}{r_2}$ is in E. Let $r = r_1 + r_2$

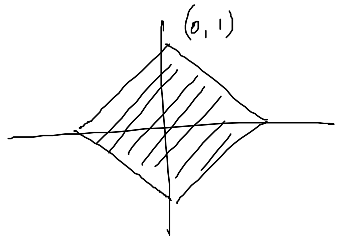
then
$$\frac{x+y}{r} = \frac{r_1 x}{r r_1} + \frac{r_2 y}{r r_2}$$
$$= \frac{r_1}{r}\left(\frac{x}{r_1}\right) + \frac{r_2}{r}\left(\frac{y}{r_2}\right) \qquad \frac{r_1}{r} + \frac{r_2}{r} = 1$$
$$\in E.$$

$$\Rightarrow \frac{x+y}{r} \in E \Rightarrow p_E(x+y) \le r = r_1 + r_2$$
$$< p_E(x) + p_E(y) + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows
$$p_E(x+y) \le p_E(x) + p_E(y)$$
as desired. //

for example ... in $\mathbb{R}^2$

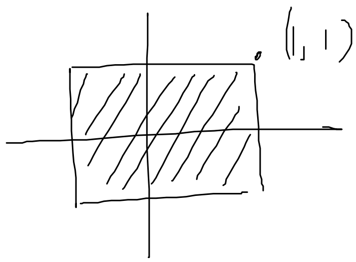
if $E =$



i.e. $S_\infty\left(\binom{0}{0}, 1\right)$

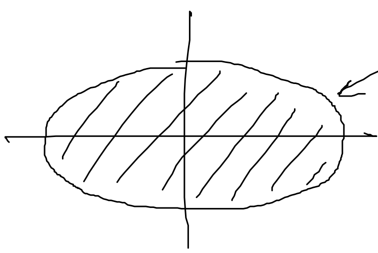
then $p_E((x,y)) = |x| + |y|$

if $E =$



then $p_E((x,y)) = \max\{|x|, |y|\}$

if $E =$



$ax^2 + by^2 \leq 1$

then $p_E((x,y)) = \sqrt{ax^2 + by^2}$