Vector Spaces

Consider a nonempty set L with two operations

$+ : L \times L \to L$ (a function)

and

$\cdot : \mathbb{R} \times L \to L$

(Note: $\cdot$ can go from $\mathbb{C} \times L \to L$ or from $K \times L \to L$ where K is any field.)

Then $(L, +, \cdot, K)$ is a vector space if

- $x + y = y + x$ commut.
- $(x+y) + z = x + (y+z)$ assoc.
- $\exists\, 0 \in L$ so that $x + 0 = x \quad \forall x \in L$
- given $x \in L$ $\exists$ $-x \in L$ so that $x + (-x) = 0$.

- $\alpha(\beta x) = (\alpha \cdot \beta) x \quad \forall \alpha, \beta \in K \quad \forall x \in L$
- $1 \cdot x = x$
- $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x \quad \forall \alpha, \beta \in K \quad \forall x \in L$
- $\alpha(x+y) = \alpha \cdot x + \alpha \cdot y \quad \forall \alpha \in K \quad \forall x, y \in L$

elements of L are called "vectors".

this is familiar in terms of $\mathbb{R}^n$

$\mathbb{R}^n$ is a vector space over $\mathbb{R}$ ($= K$).

Note: $L^\infty([a,b])$ = continuous real-valued functions on $[a,b]$

is a vector-space over $\mathbb{R}$.

X = continuous complex-valued functions on $[a,b]$ is a vector space over $\mathbb{R}$. It's also a vector space over $\mathbb{C}$.

{Infinite sequences of real numbers}

is a vector space, if we define

$\{x_n\} + \{y_n\} = \{x_n + y_n\}$

$\alpha\{x_n\} = \{\alpha x_n\}$

We'll be more concerned with infinite dimensional vector spaces than finite dimensional ones.

Which brings us to dimension.

$\{x_1, \dots x_n\}$ are linearly dependent if

$\exists\ \alpha_1, \dots \alpha_n$ so that ~~so~~ some $\alpha_i \neq 0$ and

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0.$$

If $\alpha_{i_0} \neq 0$ then $x_{i_0} = \frac{1}{\alpha_{i_0}} \left(\sum_{j \neq i_0} \alpha_j x_j \right)$

i.e. x_{i_0} is redundant — it can be written as a linear combination of the others.

---

a vector space L is n-dimensional if n linearly independent elements exist, but any set of $n+1$ elements is linearly dependent.

Subspaces are defined in a natural manner.

If $L = \{\text{polynomials}\}$ w/ $K = \mathbb{R}$

$L' = \{\text{polys of degree } 2\}$ isn't a subspace

$L' = \{\text{polys of degree} \leq 2\}$ is a subspace

Given a subspace L', you can define a factor space (or "quotient space"). You do this by first defining an equivalence relation on L:

$$x, y \in L \text{ then } x \sim y \text{ if } x - y \in L'.$$

Once you've defined the equivalence relation, you have equivalence classes

$$\{x\} = \{y \in L \mid x \sim y\}$$

we denote the collection of equivalence classes by L/L'.

You can add $\{x\}$ and $\{\tilde{x}\}$ by choosing a representative of $\{x\}$ and of $\{\tilde{x}\}$ and adding them.

$$x \text{ represents } \{x\} \iff \forall y \in \{x\} \quad x \sim y$$
$$\iff \forall y \in \{x\}, \; x - y = z \in L'$$

$$\tilde{x} \text{ repr. } \{\tilde{x}\} \iff \forall \tilde{y} \in \{\tilde{x}\} \quad \tilde{x} \sim \tilde{y}$$
$$\iff \tilde{x} - \tilde{y} = \tilde{z} \in L'$$

$$\Rightarrow x + \tilde{x} = (y + z) + (\tilde{y} + \tilde{z})$$
$$= (y + \tilde{y}) + \underbrace{(z + \tilde{z})}_{\in L'}$$

so $\{x\} + \{\tilde{x}\} = \{x + \tilde{x}\}$ makes sense.

similarly, you can multiply $\{x\}$ by $\alpha \in K$.

$\Rightarrow$ L/L' is a vector space in its own right.

<u>defn:</u> if $\dim(L/L') = n$ then we say L' has <u>codimension</u> n

<u>theorem:</u> Let L' be a subspace of vector space L. Then L' has finite codimension if and only if there are linearly indep. elements $x_1, .. x_n \in L$ s. that every element $x \in L$ has a unique representation of the form

$$x = \alpha_1 x_1 + \alpha_2 x_2 + .. + \alpha_n x_n + y$$

where $\alpha_1 .. \alpha_n \in K$ and $y \in L'$.

<u>proof:</u> see K + F

<u>linear functionals</u>

$$f : L \to K$$

is additive if $f(x+y) = f(x) + f(y)$ $\forall x, y \in L$

is <u>homogeneous</u> if

$$f(\alpha x) = \alpha f(x) \quad \forall \alpha \in K \quad \forall x \in L$$

for example

$F : L^\infty([a,b]) \to \mathbb{R}$

with $F(f) = \int_a^b f(x)\cos(x)\,dx$

is an additive, homogeneous functional

<u>defn</u>: a linear functional is an additive homogeneous functional

Let f be a linear functional on L.
we define $L_f \subset L$ by $\{x \in L_f \mid f(x) = 0\}$

clearly, L_f is a subspace.
and we'd hope it's co-dimension 1 since
0 is codimension 1 in $\mathbb{R}$. what if $\mathbb{R} = K$?

<u>theorem</u>: Let $x_0 \in L - L_f$. fix x_0. then every element $x \in L$ can be written! as

$$x = \alpha x_0 + y$$

for some $\alpha \in K$, some $y \in L_f$.

proof:

Since $x_0 \in L - L_f$, we know $f(x_0) \neq 0$.

Given $x \in L$ we define

$$y = x - \alpha x_0$$

where $\alpha := \frac{f(x)}{f(x_0)}$

then $f(y) = 0 \Rightarrow y \in L_f$.

and $x = \alpha x_0 + y$. So we've found one representation. What if

$$x = \tilde{\alpha} x_0 + \tilde{y} \quad \text{for some other } \tilde{\alpha} \in K, \tilde{y} \in L_f ?$$

then $0 = (\alpha - \tilde{\alpha})(x_0) + y - \tilde{y}$

$\Rightarrow (\alpha - \tilde{\alpha}) x_0 = \tilde{y} - y \in L_f$

$\Rightarrow (\alpha - \tilde{\alpha}) f(x_0) = 0 \Rightarrow \alpha = \tilde{\alpha}. \Rightarrow y = \tilde{y}$

$\Rightarrow$ representation is unique. //

This proves L_f has codimension 1.

Corr: $x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2)$

Corr: if f and g are two linear functionals $\ni$ $L_f = L_g$ then $f = \alpha g$ some $\alpha \in K$.

thm: let L be a vector space, f a nontrivial linear functional on L. Then

$$M_f := \{x \mid f(x) = k\}$$

is a hyperplane parallel to the null space L_f of the functional. Conversely, let $M' = L' + x_0$ be any hyperplane parallel to a subspace of codimension 1 so that $0 \notin M$. Then $\exists!$ linear functional f on L so that $M' = \{x \mid f(x) = k_0\}$. ($k_0$ fixed. $k_0 \neq 0$ take k_0 = the identity of K if you like.)

proof:

Given f, take $x_0 \ni f(x_0) = k$. (take some x_0 so that $f(x_0) \neq 0$ and then look at $x_0 \cdot \frac{k}{f(x_0)}$.) Then $x_0 \in L - L_f$ and by previous theorem, every $x \in L$ can be uniquely written as $x = \alpha x_0 + y$ some $\alpha \in K$, some $y \in L_f$. If $x \in M_f$ then we have $\alpha = 1$. $\Rightarrow$ every $x \in M_f$ can be written as $x_0 + y$ some $y \in L_f$. $\Rightarrow$ M_f is a hyperplane parallel to L_f. ✓

Now, assume

$M' = x_0 + L'$

be a hyperplane parallel to a codimension 1 subspace L'. Since L' has codimension 1, if $x \in L$ then $\exists!\, \alpha$ and $y \in L'$ so that

$x = \alpha x_0 + y$.

We define $f(x) = \alpha k_0$. the claim is that f is a unique linear functional and $L_f = L'$.

f is certainly a linear functional. we need to check

$L_f = L'$

$x \in M' \Rightarrow f(x) = k_0$

and f is unique.

if $x \in M'$ then $x = x_0 + y$ some $y \in L'$

$\Rightarrow \alpha = 1$ and $f(x) = k_0$ ✓

If $\exists$ linear functional g so that $M' = \{x \mid g(x) = k_0\}$ then $L_g = L'$ and $g = f$.

$L_g = L'$ since given $y \in L'$, $x_0 + y \in M'$

$\Rightarrow g(x_0 + y) = g(x_0) + g(y) = k_0$ by def of M'

on the other hand, $0 \in L' \Rightarrow g(x_0 + 0) = g(x_0) + g(0) = g(x_0) = k_0$

$\Rightarrow g(y) = 0$ for all $y \in L'$, and since $g(x_0) \neq 0$, $L_g \neq L \Rightarrow L' = L_g$

So that prove $L_g = L'$. Now we just want to prove $f = g$.

Let $x \in L$. then $x = \alpha x_0 + y$ some $y \in L'$

$\Rightarrow f(x) = \alpha f(x_0) + f(y) = \alpha k_0$

$g(x) = \alpha g(x_0) + g(y) = \alpha g(x_0) = \alpha k_0$

$\Rightarrow f(x) = g(x) \ \forall x \in L \Rightarrow f = g$. and the functional is unique. //