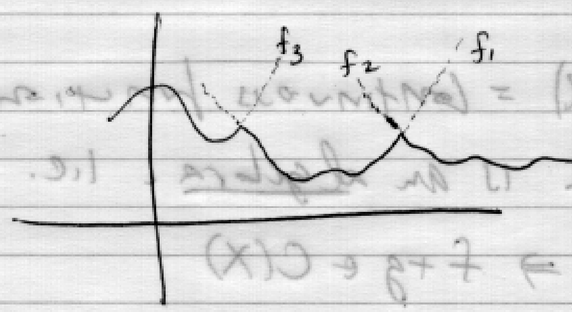
Stone Weierstrauss Theorem

We know that polynomials are dense in $L^\infty([a,b])$ where $L^\infty([a,b])$ is the space of continuous functions on $[a,b]$ with the metric
$$\rho(f,g) = \max_{x\in[a,b]} |f(x)-g(x)|.$$

If you think about $L^\infty([a,b])$, it has some very nasty functions in it. i.e. functions that are nowhere differentiable.

There are two key ideas behind the proof that polynomials are dense in $L^\infty([a,b])$.

1. Think about the graph of g



this graph looks like $\min\{f_1, f_2, f_3\}$ where f_1, f_2, f_3 are nicer than g.

idea 1: having $f_1 \wedge f_2 = \min\{f_1, f_2\}$

$f_1 \vee f_2 = \max\{f_1, f_2\}$

Will likely be useful.

<u>Idea 2:</u> $f \wedge g = \min\{f, g\}$

$$= \tfrac{1}{2}(f+g) - \tfrac{1}{2}|f-g|$$

$$f \vee g = \tfrac{1}{2}(f+g) + \tfrac{1}{2}|f-g|$$

So if we can approximate the absolute value function w/ polynomials, we'll be on our way.

Rather than restricting ourselves to polynomials, which means $f: \mathbb{R}^n \to \mathbb{R}$, we'll extend to $f: X \to \mathbb{R}$ where X is compact.

<u>observation:</u> $C(X)$ = continuous functions from $X \to \mathbb{R}$ is an <u>algebra</u>. i.e.

$f, g \in C(X) \Rightarrow f+g \in C(X)$

$a, b \in \mathbb{R}$, $f, g \in C(X) \Rightarrow af + bg \in C(X)$.

$f, g \in C(X) \Rightarrow fg \in C(X)$.

<u>defn:</u> F, a family of functions from $X \to \mathbb{R}$ <u>separates points</u> if given $x \neq y$, $\exists$ $f \in F$ such that $f(x) \neq f(y)$.

<u>defn</u>: $L \subset C(X)$ is a <u>lattice</u> if given $f, g \in L$ then $f \vee g$ and $f \wedge g \in L$ where $(f \wedge g)(x) = \min\{f(x), g(x)\}$ and $(f \vee g)(x) = \max\{f(x), g(x)\}$.

<u>Lemma 1</u>: Given $\varepsilon > 0$ $\exists$ a polynomial P in one variable such that for all $s \in [-1, 1]$ we have $|P(s) - |s|| < \varepsilon$

<u>Proof</u>: Consider $\sqrt{1-t}$. If we approximate $\sqrt{1-t}$ with $Q_N(t)$ a polynomial then $Q_N(1-s^2)$ will be an approximation of $|s|$.

Let $\sum_0^\infty c_n t^n$ be the Taylor series expansion for $\sqrt{1-t}$. It converges uniformly on $[0, 1]$

$\Rightarrow$ given $\varepsilon > 0$ $\exists N$ so that

$$\max_{t \in [0,1]} |Q_N(t) - \sqrt{1-t}| < \varepsilon$$

$$\Rightarrow \max_{s \in [-1,1]} |Q_N(1-s^2) - |s|| < \varepsilon$$

$\Rightarrow$ we have found a polynomial that is ε-close to $|s|$ on $[-1, 1]$. //

Lemma: Let X be compact and L a lattice of continuous real-valued functions on X with the following properties:

i) L separates points; if $x \neq y$ then $\exists f \in L$ with $f(x) \neq f(y)$

ii) if $f \in L$ and $c \in \mathbb{R}$ then cf and $c + f \in L$

Then given any $h \in C(X)$ and any $\varepsilon > 0$, $\exists g \in L$ such that $\rho(h, g) < \varepsilon$.

Stone-Weierstrass Theorem: Let X be compact and A an algebra of continuous real-valued functions on X that separates the points of X and contains the constant functions. Then given any $f \in C(X)$ and any $\varepsilon > 0$ there is $g \in A$ such that $\rho(f, g) < \varepsilon$. (i.e. A is dense in $C(X)$.)

Corollary: Let $X \subset \mathbb{R}^n$ be closed and bounded. Let $f \in C(X)$. Then $\exists$ a polynomial g such that $\rho(f, g) < \varepsilon$.

Proof of S-W theorem: Let $[A]$ be the closure of A in $C(X)$. Then $[A]$ is an algebra since A is an algebra. We want to prove $[A] = C(X)$. From the lemma, it suffices to show $[A]$ is a lattice. i.e. we want to show $f \wedge g$ and $f \vee g \in [A]$ if $f, g \in [A]$.

We know

$$f \wedge g = \underbrace{\tfrac{1}{2}(f+g)}_{\in [A]} - \tfrac{1}{2}|f-g| \quad , \quad f \vee g = \tfrac{1}{2}(f+g) + \tfrac{1}{2}|f-g|$$

and $f-g \in [A]$.

So if we knew $|f-g| \in [A]$, we'd be done! i.e.

we want to show $f \in [A] \Rightarrow |f| \in [A]$.

given $f \in [A]$, let $c = \max_X |f(x)|$ then

$\frac{1}{c} f \in [A]$ and $\frac{1}{c}|f(x)| \leq 1$. Fix $\varepsilon > 0$.

$\exists$ polynomial P so that

$$\left| \left|\tfrac{1}{c} f(x)\right| - P\left(\tfrac{1}{c} f(x)\right) \right| < \varepsilon \quad \forall x \in X.$$

$\Rightarrow \rho\left(\frac{1}{c}|f|, P(\frac{1}{c}f)\right) < \varepsilon$

$\Rightarrow \rho\left(|f|, cP(\frac{1}{c}f)\right) < \varepsilon c$

Now P is a polynomial and $\frac{1}{c}f \in [A]$ where $[A]$ is an algebra. $\Rightarrow P(\frac{1}{c}f) \in [A]$. $\Rightarrow$ we've found an elt of $[A]$ that is ε-close to $|f|$. $\Rightarrow |f| \in [A]$. //

Okay, so all we have to do is prove the lemma! This needs two lemmas.

<u>lemma 1</u>: Let L be a lattice in $C(X)$ and suppose the function h defined by

$$h(x) = \inf_{f \in L} f(x)$$

is continuous. Then given $\varepsilon > 0$ $\exists$ $g \in L$ so that $\rho(h, g) < \varepsilon$.

<u>proof:</u> given $x \in X$ $\exists$ $f_x \in L$ so that $h(x) < f_x(x) < h(x) + \varepsilon/3$. Since f_x and h are continuous, $\exists$ open set $O_x \ni$

$$y \in O_x \Rightarrow |f_x(x) - f_x(y)| < \varepsilon/3 \text{ and } |h(x) - h(y)| < \varepsilon/3.$$

$\Rightarrow h(y) \le f_x(y) < h(x) + \varepsilon \quad \forall y \in O_x$. Since X is compact, we can cover it with $O_{x_1}, \ldots O_{x_n}$. then if $g = f_{x_1} \wedge f_{x_2} \wedge \ldots \wedge f_{x_n}$ we have $g \in L$ and $h(x) \le g(x) < h(x) + \varepsilon \quad \forall x \in X$.

<u>lemma 2:</u> Let X compact $L \subset C(X)$ be a lattice that satisfies

(i) L separates points

(ii) if $f \in L$ and $c \in \mathbb{R}$ then cf and $c+f \in L$

Then given any $a, b \in \mathbb{R}$ and any $x, y \in X$, $x \neq y$. $\exists$ $f \in L \ni f(x) = a$ and $f(y) = b$.

<u>proof:</u> Let $g \in L \ni g(x) \neq g(y)$.

then $f = a\dfrac{g - g(y)}{g(x) - g(y)} + b\dfrac{g(x) - g}{g(x) - g(y)} \in L$

and f satisfies the condition $f(x) = a$, $f(y) = b$.

(<u>Note</u>: didn't use any algebra properties, just conditions (i) and (ii))

<u>lemma 3:</u> Let X be compact, and L a lattice in $C(X)$ that satisfies

(i) L separates points

(ii) $f \in L$ and $c \in \mathbb{R}$ then cf and $c+f \in L$.

Let $a, b \in \mathbb{R}$ with $a \leq b$, $F \subset X$ closed and $p \notin F$. Then $\exists\ f \in L$ such that $f \geq a$, $f(p) = a$, and $f(x) > b$ for all $x \in F$.

<u>proof:</u>

given $x \in F$ $\exists\ f_x \in L$ so that $f_x(p) = a$ and $f_x(x) = b+1$. (previous lemma).

Let $O_x = \{y \in X \mid f_x(y) > b\}$. Then O_x cover F which is compact. $\Rightarrow$ finitely many cover F. Let

$$f = f_{x_1} \vee f_{x_2} \vee \ldots \vee f_{x_n}.$$

then $f(p) = a$ and $f > b$ on F by construction. Since constant functions are in L (using (ii)), we know $f \vee a \in L$. then $f \vee a$ satisfies $f \vee a \in L$, $f \vee a \geq a$ on X, $(f \vee a)(p) = a$, and $f \vee a > b$ on F. //

Now we're ready to prove our key lemma.

proof: given $h \in C(X)$ we use h to define a sublattice of L:

$$L' = \{ f \in L \mid f \geq h \}.$$

we know h is continuous and if we can prove $h(p) = \inf_{L'} f(p)$ for each $p \in X$ then by lemma 1 $\exists g \in L' \subset L$ so that $\rho(h, g) < \varepsilon$, as desired.

Let $\eta > 0$. $F_\eta = \{ x \in X \mid h(p) + \eta \leq h(x) \}$.

Since h is continuous, F_η is closed. We know h is bounded on X $\ni h \leq M$. Now by the lemma 3, $\exists$ $f_\eta \in L$ such that

$f_\eta \geq h(p) + \eta$, $f_\eta(p) = h(p) + \eta$ and $f_\eta(x) > M$ on F_η.

On the (other hand) $h < h(p) + \eta$ on $X - F_\eta$. $\Rightarrow$

$h < f_\eta$ on X. $\Rightarrow f_\eta \in L'$ and

$f_\eta(p) \leq h(p) + \eta$.

$$\Rightarrow \lim_{\eta \downarrow 0} f_\eta(p) \leq h(p) \Rightarrow \inf_{L'} f(p) \leq h(p).$$

but $\inf_{L'} f(p) \geq h(p)$ by defn of L'. $\Rightarrow h(p) = \inf_{L'} f(p)$.

and done.! //