Before moving on, an interesting use of Hilbert spaces...

thm: Let $(L, \langle\cdot\rangle)$ be a Hilbert space. Let $f: L \to \mathbb{R}$ be a continuous linear function. Then $\exists\, a \in H$ so that $f(x) = \langle x, a\rangle \;\forall\, x \in L$.

proof: from before, we know that L_f is codimension 1. Let $x_0 \in L$, $x_0 \notin L_f$. Then $\mathrm{span}\{x_0\} = \{\alpha x_0 \mid \alpha \in \mathbb{R}\} = L_0$

is a closed subspace. Then since $(L, \langle\cdot\rangle)$ is a Hilbert space, we can define $L_0^\perp$ (another closed subspace) and $L = L_0 \oplus L_0^\perp$. (Note $L_f \subseteq L_0^\perp$).

Since f is continuous, L_f is closed $\Rightarrow L_f = L_0^\perp$.

Now without loss of generality $\|x_0\| = 1$ Define $a = f(x_0)\, x_0$.

$$\langle x, a\rangle = \langle h + h', a\rangle \quad \text{where } h \in L_0^\perp,\ h' \in L_0$$
$$= \langle h', a\rangle = \langle \alpha x_0, f(x_0) x_0\rangle$$
$$= \alpha f(x_0) \quad \text{since } \|x_0\| = 1$$

$$f(x) = f(h + h')$$
$$= f(h') \quad \text{since } h \in L_f$$
$$= f(\alpha x_0) = \alpha f(x_0)$$

$$\Rightarrow f(x) = \langle x, a\rangle \quad \forall\, x \in L. \;//$$

Topological Vector spaces

Given $(L, \langle\cdot\rangle)$ or $(L, \|\cdot\|)$ we have a metric and this induces a topology on τ.

A topological vector space is

$$(L, \tau)$$

where τ is a topology that respects the vector space structure. i.e. $+$ and $\cdot$ are continuous wrt the topology.

i.e. if $z_0 = x_0 + y_0$ then given an open set $z_0 \in U$ $\exists$ open sets V, W with $x_0 \in V$, $y_0 \in W$ and if $x \in V$ and $y \in W$ then $x + y \in U$. i.e. $W + V \subset U$.

Similarly if $z_0 = \alpha x_0$ then given open set $z_0 \in U$ $\exists$ open set V w/ $x_0 \in V$ and $\varepsilon > 0 \ni |r - \alpha| < \varepsilon$ $x \in V \Rightarrow rx \in U$. i.e. $rV \subset U$.

Thm: Let (L, τ) be a topological vector space. Let U be an open set containing $\vec{0}$. Then $x_0 + U = \{x_0 + y \mid y \in U\}$ is an open set containing x_0. And every open nbd of x_0 is equal to $x_0 + U$ for some open set containing $\vec{0}$.

i.e. Just need to define the topology at $\vec{0}$.

Given a local base for τ at $\vec{0}$, this local base will generate the entire topology.

proof: x_0 is fixed. $f: L \to L$ where $f(x) = x - x_0$ is cts wrt τ since the addition is cts. $\Rightarrow$ given an open set containing $\vec{0}$, $f^{-1}(U)$ is an open set.

$$0 \in U \in \tau \Rightarrow f^{-1}(U) \in \tau$$

$$f^{-1}(U) = \{x \mid x - x_0 \in U\} = x_0 + U$$

Since $f(x_0) \Rightarrow \vec{0}$, $x_0 \in f^{-1}(U) = x_0 + U$. $\Rightarrow x_0 + U \in \tau$, as claimed.

Now, let $x_0 \in V \in \tau$. I want to show $V = x_0 + U$ for some $U \in \tau$. Let $g(x) = x + x_0$. Then as before, g is continuous. $\Rightarrow g^{-1}(V) \in \tau$

$$\begin{aligned} g^{-1}(V) &= \{x \mid x + x_0 \in V\} \\ &= \{x \mid x + x_0 = y \text{ some } y \in V\} \\ &= \{x \mid x = y - x_0 \text{ some } y \in V\} \\ &= V - x_0 \end{aligned}$$

$\Rightarrow V - x_0 = U$ for some $U \in \tau$ $0 \in U$.

$\Rightarrow V = x_0 + U$ as claimed //

The idea is that since $+$ is cts and since you can get anywhere from $\vec{0}$ using $+$, this determines the topology.

Note: we've already seen this in a HW problem.

recall $X = \{$ infinitely differentiable fxns on $(-\infty, \infty)\}$

we defined a topology using the local bases

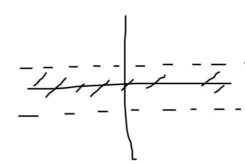
$$N(\phi; \varepsilon, k, R) = \{ \psi \in X \mid \sup_{|x| \le R} |\phi(x) - \psi(x)| < \varepsilon, \; \sup_{|x| \le R} |\phi'(x) - \psi'(x)| < \varepsilon, \; \ldots, \; \sup_{|x| \le R} |\psi^{(k)}(x) - \phi^{(k)}(x)| < \varepsilon \}.$$

This makes X (a vector space) into a topological vector space.

Q: when does (L, τ) come from $(L, \|\cdot\|)$?
i.e. when is a topological vector space normable?

defn: $M \subset L$ where (L, τ) is bounded if given any neighborhood U of $\vec{0}$ $\exists \alpha \ni$
$M \subset \alpha U = \{ \alpha z \mid z \in U \}$.

(Note: if $L = \mathbb{R}^2$ and the open sets are given by the base at $\vec{0}$ of



$$V_\varepsilon = \{ (x, y) \mid |y| < \varepsilon \}$$

then a set M can be bounded in the above sense w/o being bounded in our metric sense. You can check that $+, \cdot$ are cts.

<u>defn</u>: (L,τ) is <u>locally bounded</u> if it contains at least one nonempty bounded open set.

<u>thm</u>: $(L,\|\cdot\|)$ is a locally bounded topological vector space

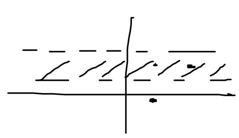
<u>proof</u>: fix $\varepsilon>0$. $\{x \mid \|x\|<\varepsilon\}$ is bounded and nonempty and open.

<u>defn</u>: (L,τ) is <u>locally convex</u> if every nonempty open set contains a nonempty convex open set.

<u>thm</u>: $(L,\|\cdot\|)$ is locally convex

<u>proof</u>: any open set contains a ball and balls are convex, nonempty, and open //

<u>thm</u>: if (L,τ) is T_1 and locally convex and locally bounded, then (L,τ) is normable.

<u>note</u>: My example of  is not T_1 since we cannot find

$$V, U \ni (1,1)\in V \quad (2,1)\notin V$$
$$(1,1)\notin U \quad (2,1)\in U.$$

(L,τ) is pretty tricky!! Work the problems in K+F!!

Let $-X = \{-x \mid x \in X\}$.
then if U is open then $-U$ is open

Also, if X and Y are two subsets of L and X is open then $X+Y$ is open.
(proof: fix $y \in Y$. then $y + X \in \tau \Rightarrow \bigcup_y y + X \in \tau \Rightarrow X + Y \in \tau$)

Since mult is cts, given $U \in \tau$, $\vec{0} \in U$, $\exists\, U' \in \tau$, $\vec{0} \in U'$ so that $U' + U' \subset U$. By induction, $\exists\, U_n \in \tau$ $\vec{0} \in U_n$ so that $\underbrace{U_n + U_n + U_n + \dots + U_n}_{n \text{ times}} \subset U$.

Since $\vec{0} \in U \in \tau \Rightarrow -U \in \tau \Rightarrow U \cap (-U) \in \tau$ we can assume there's a symmetric open set inside every open set.
$\Rightarrow$ given $\vec{0} \in U \in \tau$ $\exists$ $U_n \in \tau$, $\vec{0} \in U_n$, U_n symmetric so that

$$\underbrace{U_n + U_n + \dots + U_n}_{n \text{ times}} \subset U.$$

Lemma: Given a compact subset K in (L, τ) and a closed subset C, if $K \cap C = \phi$ then $\exists$ open nbd of $\vec{0}$, $U \in \tau$ so that $[K+U] \cap (C+U) = \phi$

Note: Since $K+U \subset [K+U]$ and since $K+U$ and $C+U$ are open sets that contain $K+C$ this is a separation result!!

Corr: If isolated points are closed in (L,τ) then (L,τ) is Hausdorff.

proof: let $x, y \in L$, $x \neq y$. Then $\{x\}$ is compact and $\{y\}$ is closed. By lemma, $\exists$ nbd of $\vec{0}$, $U \in \tau \ni$

$$[x+U] \cap y+U = \phi$$

$\Rightarrow (x+U) \cap (y+U) = \phi \Rightarrow$ have found a pair of disjoint open sets that separate $x + y$. //

Corr: let $E \subset L$. Let $\{U\}$ be a local basis of the topology at $\vec{0}$. Then

$$[E] = \bigcap_U (E+U)$$

proof: Let $C = [E]$ then C is closed. Take $x \notin C$. Then $\{x\}$ is compact. $\Rightarrow$ by lemma, $\exists$ $U \in \tau$, $\vec{0} \in U$ so that $(x+U) \cap ([E]+U) = \phi$. Since $(E+U) \subset [E]+U$, we have $(x+U) \cap (E+U) = \phi$. Since U is open, $\exists$ $\tilde{V} \in$ local base $\ni$ $0 \in \tilde{V} \subset U \Rightarrow (x+\tilde{V}) \cap (E+\tilde{V}) = \phi \Rightarrow (x+\tilde{V}) \cap \left[\bigcap_U (E+U)\right] = \phi$

therefore we've shown.

1. if $x \notin [E]$ then $x \notin \bigcap_U (E+U)$

$\Rightarrow \bigcap_U (E+U) \subseteq [E]$.

2. Now I want to show $\bigcap_U (E+U)$ is closed.

Take $x \notin \bigcap_U (E+U)$. if $x \notin [E]$ then we're done by the prev. argument. Assume $x \in [E]$. then for any open set containing x, the open set intersects E. Let $V \in$ local base at 0, then $-V$ is open $\Rightarrow$ $x - V \cap E \neq \phi$ $\Rightarrow x \in (E+V)$. Since V was an arb. member of the local base at 0, this shows

$x \in \bigcap_U (E+U)$ $\cancel{X}$ (Note: what I just showed is really $[E] \subseteq \bigcap_U (E+U)$ ...) //

<u>proof of lemma concerning K compact, C closed:</u>

Let $x \in K$ Since C is closed, $\exists$ nbd of $\vec{0}$ so that $(x+U) \cap C = \phi$. By continuity of addition, and by the symmetrization, $\exists$ open nbd of $\vec{0}$ so that $N = -N$, N open, and

$0 \in N+N+N \subset U$.

$\Rightarrow \left(x + (N+N+N)\right) \cap C = \phi$

$\Rightarrow \left(x + (N+N)\right) \cap (C-N) = \phi$

$\Rightarrow \left(x + (N+N)\right) \cap (C+N) = \phi$

Denote N by N^x. So for each $x \in K$, we've found open nbd of $\vec{0}$ (watch out for the abuse of notation here!) so that

$$\left(x + (N^x + N^x)\right) \cap (C + N^x) = \phi.$$

Since K is compact, $\exists\ x_1 \dots x_n \ni$

$$K \subset (x_1 + N^{x_1}) \cup (x_2 + N^{x_2}) \cup \dots \cup (x_n + N^{x_n})$$

let $V = \bigcap_1^n N^{x_i} \in \tau$, $\vec{0} \in V$.

then $(K+V) \subseteq \bigcup_1^n (x_i + N^{x_i} + V)$

$\subseteq \bigcup_1^n (x_i + N^{x_i} + N^{x_i})$

and $(x_i + N^{x_i} + N^{x_i}) \cap (C + N^{x_i}) = \phi$

each $x_i \Rightarrow (x_i + N^{x_i} + N^{x_i}) \cap (C+V) = \phi$

$\Rightarrow (K+V) \cap (C+V) = \phi$ as desired //

defn: $E \subset L$ is **balanced** if for every α w/ $|\alpha| \le 1$ we have

$$\alpha E \subseteq E.$$

Note: balanced is stronger than symmetric!
(symmetric: $-E = E$)

lemma: let $\vec{0} \in U \in \tau$. Then $\exists$ balanced neighborhood N of $\vec{0}$ $\ni$ $\vec{0} \in N \subset U$.

proof: Since $\cdot : K \times L \to L$ is continuous given $\alpha \in K$ and $x_0 \in L$ and open nbd of U αx_0, then $\exists\, \varepsilon > 0$ and V open nbd of x_0 so that

$$|r - \alpha| < \varepsilon, \quad x \in V$$
$$\Rightarrow \quad rx \in U$$

Specifically, mult is cts at $\alpha = 0$, $x_0 = 0$ $\Rightarrow$
$\exists\, \varepsilon > 0$ and nbd V of 0 $\ni$

$$|r - 0| = |r| < \varepsilon, \quad x \in V \Rightarrow rx \in U.$$

now fix α with $|\alpha| < \varepsilon$. then $\alpha V \subset U$.

$$\Rightarrow \bigcup_{\alpha,\, |\alpha| < \varepsilon} (\alpha V) \subset U$$

and $W = \bigcup_{\alpha,\, |\alpha| < \varepsilon} (\alpha V)$ is an balanced open set //

<u>lemma</u>: given a convex neighborhood U of $\vec{0}$, it contains V a convex balanced nbd of $\vec{0}$.

<u>defn</u>: $E \subset L$ is <u>absorbing</u> if given $x \in L$ $\exists\, t \in \mathbb{R}$ (or $\mathbb{C}$) so that

$$x \in \alpha E \quad \forall \alpha \text{ with } |\alpha| \geq t.$$

<u>lemma</u>: Let $\vec{0} \in U \in \tau$ be a neighborhood of $\vec{0}$. then $\exists$ V a neighborhood of $\vec{0}$, $V \subset U$ so that V is <u>absorbing</u>

<u>proof</u>: 1) first, assume U is balanced (if U isn't, then $\exists$ balanced $W \subset U$ and we'll put V inside W.)

fix $x \in L$. The map $f: \mathbb{R} \to L$ given by $f(\alpha) = \alpha x$ is a continuous map. $\Rightarrow$ it's continuous at $\alpha = 0$ $\Rightarrow$ $\exists\, \delta > 0 \ni$

$|\alpha - 0| < \delta \Rightarrow \alpha x \in U$. Since $\mathbb{R}$ isn't discrete, $\exists\, \alpha_0 \neq 0$ $|\alpha_0| < \delta \Rightarrow \alpha_0 x \in U \Rightarrow x \in \frac{1}{\alpha_0} U$.

and if $|\beta| \leq |\alpha_0| \Rightarrow \left|\frac{\beta}{\alpha_0}\right| \leq 1 \Rightarrow$

$$\left(\frac{\beta}{\alpha_0}\right) U \subset U \Rightarrow \beta x \in U \Rightarrow x \in \frac{1}{\beta} U.$$

$\Rightarrow x \in \gamma U \quad \forall \gamma$ with $|\gamma| \geq \frac{1}{\alpha_0} \Rightarrow U$ is absorbing. //

cts linear functionals

given $f : (L, \tau) \to \mathbb{R}$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

we say f is continuous at x_0 if given $\varepsilon > 0$
$\exists\ x_0 \in U \ni$ if $x \in U$ then

$$|f(x) - f(x_0)| < \varepsilon.$$

f is continuous on L if f is continuous at every $x_0 \in L$.

thm: Let f be a linear functional on (L, τ).
If f is continuous at $x_0 \in L$ then f is continuous at $x_1 \in L$. for all x_1. i.e. if f is continuous at one point then it is continuous everywhere.

proof: Fix $\varepsilon > 0$. Then $\exists\, x_0 \in U \in \tau$ so that

$$|f(U) - f(x_0)| < \varepsilon. \quad \text{i.e. } y \in U \Rightarrow |f(y) - f(x_0)| < \varepsilon.$$

Now, given x_1, define

$$V = U + (x_1 - x_0) = \{ z + x_1 - x_0 \mid z \in U \}$$

we know $V \in \tau$ and $x_1 \in V$.
Let $y \in V$. then $y = z + x_1 - x_0$ some $z \in U$

$$\Rightarrow f(y) = f(z) + f(x_1) - f(x_0) \Rightarrow |f(y) - f(x_1)| = |f(z) - f(x_0)| < \varepsilon. \checkmark$$

//

<u>thm:</u> Let f be a linear functional on (L, τ). Then f is cts on $L \Leftrightarrow f$ is bounded on some neighborhood of $\vec{0}$.

<u>proof:</u>

$(\Rightarrow)$ Since f is linear, $f(\vec{0}) = 0$.

Since f is cts at $\vec{0}$, given $\varepsilon > 0$ $\exists\, U \in \tau$ so that $0 \in U$ and $|f(x) - f(0)| = |f(x)| < \varepsilon$ $\forall x \in U$.

$\Rightarrow f$ is bounded on U.

$(\Leftarrow)$ assume f is bounded on some neighborhood of $\vec{0}$. i.e. $\exists\, U \in \tau$ $\vec{0} \in U$ so that $|f(x)| < C$ $\forall x \in U$. Given $\varepsilon > 0$, let

$$V = \frac{\varepsilon}{C} U = \left\{ \frac{\varepsilon}{C} x \mid x \in U \right\}.$$

then $y \in V \Rightarrow y = \frac{\varepsilon}{C} x$ some $x \in U$

$$\Rightarrow \frac{C}{\varepsilon} y \in U \Rightarrow \left| f\left(\frac{C}{\varepsilon} y\right) \right| < C \Rightarrow |f(y)| < \frac{\varepsilon}{C} \cdot C = \varepsilon$$

for all $y \in V$. $\Rightarrow f$ is cts at $\vec{0}$ //

<u>thm:</u> Let f be a linear functional on (L, τ). If f is continuous on L then f is bounded on every bounded set. If (L, τ) is first countable and f is bounded on every bounded set, then f is cont.

proof:

1) assume f is cts. Show f is bounded on all bounded set. Let $M \subset L$ be a bounded set. Since f is cts at $\vec{0}$, $\exists$ $U \in \tau$ so that $\vec{0} \in U$ and f is bounded on U. Since M is bounded, $\exists$ $\alpha \ni M \subset \alpha U$. $\Rightarrow$ if $y \in M$ then $y = \alpha x$ some $x \in U$ $\Rightarrow |f(y)| = |f(\alpha x)| = |\alpha f(x)| \leq |\alpha| C$
$\Rightarrow f$ is bounded on M.

2) Assume (L, τ) is first countable. Then $\exists$ a countable base for the topology at 0. i.e. $\{U_k\}_1^\infty \subset \tau$
$\Rightarrow$ given $U \in \tau$ $\vec{0} \in U$ then $U_k \subseteq U$ some U_k.
Moreover, we can assume $U_1 \supset U_2 \supset U_3 \supset \ldots$
(just take $\tilde{U}_k = \bigcap_1^k U_n$.) If f is not continuous at $\vec{0}$ then $\nexists$ open set U so that f is bounded on U. $\Rightarrow$ f not bounded on any of the U_k. $\Rightarrow$ given U_k $\exists$ $x_k \in U_k$ w/ $|f(x_k)| > k$.
We know $\{x_k\}$ is bounded because
given U open nbd of $\vec{0}$, U contains V an absorbing nbd $\Rightarrow$
V contains $\{x_k\}$ for $|\alpha| \geq t_0$ $\Rightarrow$ αU contains $\{x_k\}$ for $|\alpha| \geq t_0$.
we have a bounded set which f is not bounded on $\Rightarrow f$ is not bounded on every bounded set. //