Recall the <u>Reisz</u> - <u>Fischer</u> <u>theorem</u>:

Given $\{\phi_k\}$ orthonormal in a complete $(L, \langle \cdot \rangle)$. Let $c_1, c_2 \ldots \in \mathbb{R}$ (or $\mathbb{C}$) such that

$$\sum_1^\infty |c_k|^2 < \infty.$$

Then $\exists\, f \in L$ so that

1) $c_k = \langle f, \phi_k \rangle$
2) $\sum_1^\infty |c_k|^2 = \|f\|^2$.

<u>defn</u>: $(L, \langle \cdot \rangle)$ is a <u>Hilbert Space</u> if it is complete, separable, and infinite dimensional.

<u>ex</u>: $\ell^2(\mathbb{R}, \mathbb{N})$ with $\langle x, y \rangle = \sum_1^\infty x_i y_i$.

<u>ex</u>: "integrable" fxns on $[a,b]$ that satisfy $\int_a^b |f(x)|^2 dx < \infty$ w/ $\langle f, g \rangle = \int_a^b f(x) \overline{g(x)}\, dx$

(<u>recall</u>: the space of cts fxns w/ this inner product is not complete. Its completion is the space of L^2-integrable functions ... more later.)

<u>thm</u>: any two Hilbert spaces are isomorphic.

<u>proof</u>: given $(L, \langle, \rangle)$ and $(\tilde{L}, \langle\langle, \rangle\rangle)$ we want to find a 1:1 correspondence between L and $\tilde{L}$ $x \leftrightarrow \tilde{x}$ so that the vector space and inner

product is respected, i.e

$L \leftrightarrow \tilde{L}$ is 1:1 and onto and

if $x \leftrightarrow \tilde{x}$, $y \leftrightarrow \tilde{y}$ then $x+y \leftrightarrow \tilde{x}+\tilde{y}$, $\alpha x \leftrightarrow \alpha \tilde{x}$,

and $\langle x, y\rangle = \langle\langle \tilde{x}, \tilde{y} \rangle\rangle$.

Riesz-Fischer is our friend. We'll prove that $\exists$ such a correspondence between $(L, \langle\,\rangle)$ and $(\ell^2, \langle , \rangle)$. Then by symmetry, $\exists$ correspondence between $(\tilde{L}, \langle\langle , \rangle\rangle)$ and $(\ell^2, \langle , \rangle)$ and we're done by transitivity.

Given $(L, \langle , \rangle)$ a Hilbert space, let $\{\phi_k\}_1^\infty$ be a complete orthonormal system in L. ($\exists$ because L is separable). Given $f \in L$, use $\{\phi_k\}$ to find $c_k \in \ell^2$ $(c_k = \langle f, \phi_k\rangle)$. We know $c_k \in \ell^2$ because complete $\Rightarrow$ closed $\Rightarrow$ $\|f\|^2 = \sum_1^\infty |c_k|^2$.

This defines the correspondence from L to ℓ^2. We need to show it is 1:1 and onto and respects the vector and inner product space structure.

<u>onto:</u> Given $\{c_n\} \in \ell^2$, by Riesz-Fischer theorem, $\exists f \in L$ w/ $\langle f, \phi_n\rangle = c_n$. (since L is complete)

1:1: assume $f \to \{c_k\}$, $g \nearrow$

then $\langle f, \phi_k \rangle = c_k = \langle g, \phi_k \rangle \quad \forall k$

$\Rightarrow \langle f-g, \phi_k \rangle = 0 \quad \forall k \Rightarrow f \equiv g$ because $\{\phi_k\}$ is compl.

respect $+, \cdot$!

$$\begin{aligned} \langle f, \phi_k \rangle &= c_k \\ \langle \tilde{f}, \phi_k \rangle &= \tilde{c}_k \end{aligned} \Rightarrow \langle f+\tilde{f}, \phi_k \rangle = \langle f, \phi_k \rangle + \langle \tilde{f}, \phi_k \rangle = c_k + \tilde{c}_k \checkmark$$

$$\langle \alpha f, \phi_k \rangle = \alpha \langle f, \phi_k \rangle = \alpha c_k = \checkmark$$

inner product:

$$\langle f, f \rangle = \sum_1^\infty c_k^2 \qquad \langle \tilde{f}, \tilde{f} \rangle = \sum_1^\infty \tilde{c}_k^2$$

$$\Rightarrow \langle f+\tilde{f}, f+\tilde{f} \rangle = \sum_1^\infty (c_k + \tilde{c}_k)^2 = \sum_1^\infty c_k^2 + 2 c_k \tilde{c}_k + \tilde{c}_k^2$$

$$\parallel$$

$$\langle f, f \rangle + 2\langle f, \tilde{f} \rangle + \langle \tilde{f}, \tilde{f} \rangle$$

$\Rightarrow \langle f, \tilde{f} \rangle = \sum_1^\infty c_k \tilde{c}_k$ and inner product is preserved. Note I used L as real vector space. You need to check the case if L is a complex vector space! //

Recall that if we have an inner product, then we have a norm and a metric. And we mean "subspace" = smallest closed set that contains the thing we called a subspace back in linear algebra.

<u>lemma</u>: if (X, ρ) has a countable dense subset A and $X_0 \subset X$, then $\exists$ a countable dense subset $A_0 \subset X_0 \ni [A_0] = X_0$.

<u>proof</u>: see K+F. The real use of the lemma at the moment is that if $(L, \langle . \rangle)$ is separable and L_0 is a subspace of L then L_0 is also separable.

<u>thm</u>: if L_0 is a subspace of L and $(L, \langle , \rangle)$ is a Hilbert space then either $(L_0, \langle , \rangle)$ is a Hilbert space or $(L_0, \langle . \rangle)$ is complete, separable but not infinite dimensional

<u>proof</u>: Completeness is inherited by L_0 because it's closed. Separability is inherited by above lemma. Inner product and vector space structure inherited by defn. Only question: is L_0 ∞ dimensional. If yes, then $(L_0, \langle . \rangle)$ is a Hilbert space. If no, then it's just complete + separable. //

thm: Let L_0 be a subspace of $(L, \langle \cdot \rangle)$ where $(L, \langle \cdot \rangle)$ is a Hilbert space. Define

$$L_0^{\perp} = L \ominus L_0$$

to be the elts of L that are orthogonal to L_0. Then $L_0^{\perp}$ is a subspace of $(L, \langle \cdot \rangle)$

proof - Need to show $L_0^{\perp}$ is a subspace and that it's closed.

Subspace: assume $x, y \in L_0^{\perp}$ then $\langle x, z \rangle = 0 = \langle y, z \rangle$ $\forall z \in L_0$. $\Rightarrow \langle \alpha x, z \rangle = \alpha \langle x, z \rangle = 0$

$$\Rightarrow \alpha x \in L_0^{\perp}$$

$$\Rightarrow \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle = 0 + 0 = 0$$

$$\Rightarrow x + y \in L_0^{\perp}$$

$\Rightarrow L_0^{\perp}$ is a "linear manifold". Need to show $L_0^{\perp}$ is closed. assume $x_n \in L_0^{\perp}$, $x_n \to x_0$, want $x_0 \in L_0^{\perp}$.

$$\lim_{n \to \infty} \langle x_n, z \rangle = 0 \quad \text{since } x_n \in L_0^{\perp}$$

$$\shortparallel$$

$$\langle \lim_{n \to \infty} x_n, z \rangle = \langle x_0, z \rangle \Rightarrow x_0 \in L_0^{\perp}$$

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Note: $(L_0^{\perp})^{\perp} = L_0$. L_0, and $L_0^{\perp}$ are mutually orthogonal subspaces of $(L, \langle \cdot \rangle)$

thm: Let L_0 be a subspace of a Hilbert space $(L, \langle\ \rangle)$. then every elt $f \in L$ has a unique representation as

$$f = h + h'$$

where $h \in L_0$, $h' \in L_0^{\perp}$. (i.e. $L = L_0 \oplus L_0^{\perp}$.)

proof: given $f \in L$, then $\{\phi_n\}_1^\infty$ be an orthonormal basis in L_0 (exists since $(L_0, \langle\cdot\rangle)$ is a Hilbert space.)

Let $h = \sum_1^\infty c_n \phi_n$ where $c_n = \langle f, \phi_n \rangle$

we know $\|h\|^2 = \sum_1^\infty c_n^2 < \infty$ and by Riesz-Fischer, $h \ni$ $h \in L_0$. Now define $h' = f - h$.

we know $\langle h', \phi_n \rangle = 0 \quad \forall n$

Since $z \in L_0 \Rightarrow z = \sum_1^\infty a_k \phi_k$ this gives us $\langle h', z \rangle = 0$ $\forall z \in L_0 \Rightarrow h' \in L_0^{\perp}$.

$$\Rightarrow f = \underset{\in L_0}{h} + \underset{\in L_0^{\perp}}{h'}$$

Now, uniqueness. Assume $f = \tilde{h} + \tilde{h}'$ too. $\Rightarrow \langle h, \phi_k \rangle = \langle \tilde{h}, \phi_n \rangle$ $\forall k \Rightarrow h = \tilde{h}$ because $\{\phi_k\}$ is complete in L_0. $\Rightarrow \tilde{h}' = h'$. done! //

<u>Corrollaries</u> see K+F.

<u>Q</u>: given $(L, \|\cdot\|)$ when do you know if $\| \|$ comes from $(L, \langle\cdot\rangle)$? <u>the Parallelogram rule</u>

<u>thm</u>: $(L, \|\cdot\|)$ comes from $(L, \langle\cdot\rangle) \iff$

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2)$$

$\forall f, g \in L$.

<u>proof</u>.

$(\Rightarrow)$ $\|f+g\|^2 + \|f-g\|^2 = \langle f+g, f+g\rangle + \langle f-g, f-g\rangle$

$$= \langle f, f\rangle + 2\langle f, g\rangle + \langle g, g\rangle + \langle f, g\rangle - 2\langle f, g\rangle + \langle g, g\rangle$$

$$= 2\|f\|^2 + 2\|g\|^2.$$

(<u>Note</u>: the theorem also applies when $(L, \langle\cdot\rangle)$ is a <u>complex</u> inner prod. space --- check it yourself!!)

$(\Leftarrow)$ define

$$\langle f, g\rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2\right)$$

check that $\langle\cdot,\cdot\rangle$ has all the properties of an inner product. See K+F! //

<u>Corr</u>: ℓ^p is not a hilbert space for $p \in [1, \infty]$, $p \neq 2$