Let $\{\phi_i\}$ be an orthonormal system in $(L, \langle \cdot, \cdot \rangle)$. Then $c_k = \langle f, \phi_k \rangle$ is the <u>k-th</u> <u>Fourier</u> <u>coefficient</u> of f with respect to $\{\phi_k\}_1^\infty$.

And $\sum_1^\infty c_k \phi_k$ is called the <u>Fourier Series</u> of f with respect to $\{\phi_k\}$.

A natural question is: does $f = \sum_1^\infty c_k \phi_k$?
Certainly we need some rules since if $\{\phi_k\} = \{\sin(kx)\}_0^\infty$ on $[-\pi, \pi]$ then this is orthonormal (after normalization) but if f isn't an odd function then

f will never equal $\sum_1^\infty c_k \sin(kx)$.

<u>Note</u>: "Fourier coefficient" is defined for <u>any</u> orthonormal family, not just the trigonometric functions.

eg. if $f \in C([-1, 1])$ with
$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) w(x)\, dx$$
where $w(x) = \sqrt{1-x^2}$ then $\{\phi_k(x)\}$ can be the Chebyshev polynomials

or if $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$ $\quad (w(x) = 1)$
then $\{\phi_n\}$ could be the Legendre polynomials

In general, if $w(x) \geq 0$ almost everywhere and cont.
then $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) w(x)\, dx$ is an inner
product and we can find an orthonormal
family by orthogonalizing $\{x^k\}_0^\infty$.

Q: given an orthonormal family $\{\phi_k\}_1^\infty$,
how can we find the closest fit to f
using $\phi_1 \ldots \phi_n$?

Answer: if $f \in L$ then

$$\| f - \sum_1^n a_k \phi_k \|^2 \text{ will be minimized}$$

when $a_k = \langle f, \phi_k \rangle =: c_k$. It will have
the minimum value of $\|f\|^2 - \sum_1^n c_k^2$

and $\lim_{n \to \infty} \sum_1^n c_k^2 = \sum_1^\infty c_k^2 \leq \|f\|^2$.

proof: Let $S_n = \sum_1^n a_k \phi_k$

$$\| f - S_n \|^2 = \langle f - \sum_1^n a_k \phi_k, f - \sum_1^n a_k \phi_k \rangle$$

$$= \langle f, f \rangle - 2\sum_{1}^{n} a_k c_k + \sum_{1}^{n} a_k^2$$

$$= \|f\|^2 - \sum_{1}^{n} c_k^2 + \sum_{1}^{n} (a_k - c_k)^2$$

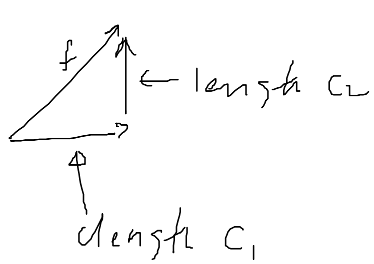
$\Rightarrow$ to minimize $\|f - S_n\|^2$ we want to make $\sum_{1}^{n} (a_n - c_n)^2 = 0$ i.e. $a_n = c_n$. Now, when $a_n = c_n$, $\|f - S_n\|^2 = \|f\|^2 - \sum_{1}^{n} c_n^2$ as claimed.

$$\Rightarrow \quad 0 \le \|f\|^2 - \sum_{1}^{n} c_n^2 \quad \Rightarrow \quad \sum_{1}^{n} c_n^2 \le \|f\|^2$$

$\Rightarrow$ $\sum_{1}^{\infty} c_n^2$ exists and $\sum_{1}^{\infty} c_n^2 \le \|f\|^2$, done! //

This shows that if we project onto an orthonormal family then the sum of the squares of the lengths of the projections cannot exceed the length (squared) of the original object.

In 2-d:



$$c_1^2 + c_2^2 \le f^2$$

(equals f^2 if all dimensions represented.)

<u>defn:</u> if for each $f \in L$ we have

$$\sum_1^\infty c_k^2 = \|f\|^2$$

then we say $\{\phi_k\}$ is a <u>closed</u> <u>orthonormal</u> system

(Oh goody! Yet another defn that uses the word closed.)

From before, $\{\phi_k\}_1^\infty = \{\sin(kx)\}$ is not a closed orthonormal system for cts real-valued functions on $[-\pi, \pi]$ w/ inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$

because if $f \equiv 1$ then $c_k = 0 \ \forall k$
and $\|f\|^2 \neq 1 \Rightarrow \sum_1^\infty c_k^2 \neq \|f\|^2$

From before, $\{\phi_n\}$ is complete if given $f \in L$, given $\varepsilon > 0$, $\exists\, a_1 \ldots a_n \ni$

$$\|f - \sum_1^n a_k \phi_k\| < \varepsilon.$$

How does $\{\phi_k\}$ complete relate to $\{\phi_k\}$ closed? Just because $\sum_1 c_k^2 = \|f\|^2$ does that imply $\sum_1^\infty c_k \phi_k = f$?!?

__theorem:__ $\{\phi_k\}$ an orthonormal family in $(L, \langle\cdot,\cdot\rangle)$ is closed $\iff$ $\sum_{1}^{\infty} c_k \phi_k = f$.

__proof:__ see K+F.

__theorem:__ $\{\phi_k\}$ an orthonormal family in $(L, \langle\cdot,\cdot\rangle)$ is complete $\iff$ $\{\phi_k\}$ is closed

__Proof:__ see K+F

__corr:__ if $(L, \langle\cdot,\cdot\rangle)$ is separable then $\exists$ a closed orthonormal system.

__proof:__ Since $(L, \langle\cdot,\cdot\rangle)$ is separable, $\exists$ a complete orthonormall system. From above, this complete system is closed. //

__Application:__ From your HW, if f is a complex-valued continuous function on $[0, 2\pi]$ then $\{e^{ikx}\}_{-\infty}^{\infty}$ is a complete orthonormal family. Since the cts fxns on $[0, 2\pi]$ are separable, $\{e^{ikx}\}_{-\infty}^{\infty}$ is a closed family

$$\Rightarrow \sum_{-\infty}^{\infty} c_k e^{ikx} = f(x) \quad \text{and} \quad \|f\|^2 = \sum_{-\infty}^{\infty} |c_k|^2$$

$$\langle f, g\rangle = \int_0^{2\pi} f(x)\,\overline{g(x)}\,dx$$

Assume $\{f_n\}$ is a sequence of functions so that $f_n, \frac{\partial f_n}{\partial x}, \frac{\partial^2 f_n}{\partial x^2}$ are cts on $[0, 2\pi]$. And $f_n \to 0$ in L^2 and $\|f_{nxx}\|^2$ bounded. Prove $\|f_{nx}\|^2 \to 0$.

we know
$$f_n = \sum_{-\infty}^{\infty} c_{nk} e^{ikx} \qquad \|f_n\|^2 = \sum_{-\infty}^{\infty} |c_{nk}|^2$$
$$f_{nx} = \sum_{-\infty}^{\infty} ik c_{nk} e^{ikx} \qquad \|f_{nx}\|^2 = \sum_{-\infty}^{\infty} |k|^2 |c_{nk}|^2$$
$$f_{nxx} = \sum_{-\infty}^{\infty} -k^2 c_{nk} e^{ikx} \qquad \|f_{nxx}\|^2 = \sum_{-\infty}^{\infty} |k|^4 |c_{nk}|^2$$

$$\Rightarrow \|f_{nx}\|^2 = \sum_{-\infty}^{\infty} |k|^2 |c_n| \cdot |c_n|$$
$$\leq \sqrt{\sum_{-\infty}^{\infty} |k|^4 |c_{nk}|^2} \sqrt{\sum_{-\infty}^{\infty} |c_{nk}|^2}$$
$$= \|f_{nxx}\|^2 \, \|f_n\|^2$$
$$\underset{\text{bounded}}{\downarrow} \qquad \underset{0}{\downarrow}$$

$$\Rightarrow \|f_{nx}\|^2 \to 0 \text{ as } n \to \infty.$$

Using $\{e^{ikx}\}_{-\infty}^{\infty}$ we can define fractional derivatives --- f "has 7.2 derivatives" if $f = \sum_{-\infty}^{\infty} c_k e^{ikx}$ and $\sum_{-\infty}^{\infty} |k|^{14.4} |c_n|^2 < \infty$.

Q: given $\{c_k\} \ni \sum_1^\infty c_k^2 < \infty$, does there exist $f \in L$ so that

1) $c_k = \langle f, \phi_k \rangle$

2) $\sum_1^\infty c_n^2 = \|f\|^2$?

A: Yes, if $(L, \langle \cdot \rangle)$ is complete!

Proof: Define $f_n = \sum_1^n c_k \phi_k$.

then certainly $\langle f_n, \phi_k \rangle = c_k$ if $k \le n$.

want $f_n \to f$ as $n \to \infty$ and for f to satisfy $\langle f, \phi_k \rangle = c_k$

$$\|f_{n+p} - f_n\|^2 = \|c_{n+1}\phi_{n+1} + c_{n+2}\phi_{n+2} + \dots + c_{n+p}\phi_{n+p}\|^2$$
$$= \sum_{n+1}^{n+p} c_n^2 < \varepsilon \quad \text{if} \quad n \geq N_\varepsilon.$$

$\Rightarrow \{f_n\}$ is a cauchy sequence and since L is complete, $f_n \to f \in L$.

$$\langle f, \phi_k \rangle = \langle f_n, \phi_k \rangle + \langle f - f_n, \phi_k \rangle = c_k + \langle f - f_n, \phi_k \rangle$$
if $n > k$

$\Rightarrow |\langle f, \phi_n \rangle - c_n| = |\langle f - f_n, \phi_k \rangle| \le \|f - f_n\| \, \|\phi_k\| \to 0$

$\therefore \langle f, \phi_n \rangle \to 0$ as $n \to \infty$.

Note: we actually proved something stronger

we proved $\langle f, \phi_k \rangle = c_k$

since the LHS is indep of n.

finally, $\langle f - \sum_1^n c_k \phi_k, f - \sum_1^n c_k \phi_k \rangle$

$$= \langle f, f \rangle - \sum_1^n c_k^2 \to 0$$

as $n \to \infty$ $\Rightarrow \sum_1^\infty c_k \phi_k = f$, as desired. //

thm: Let $\{\phi_k\}$ be an orthonormal system in a complete space $(L, \langle , \rangle)$. Then $\{\phi_k\}$ is complete $\Leftrightarrow$ L contains no elt that is orthogonal to all the ϕ_k.

(Note: already saw this w/ the $\{\sin(kx)\}_1^\infty$ example)

proof:

$(\Rightarrow)$ assume $\{\phi_k\}$ is complete. Then $\{\phi_k\}$ is closed. Assume $\langle f, \phi_k \rangle = 0$ $\forall k$. Then $c_k = 0$ $\forall k$. Since $\{\phi_k\}$ is closed, $\|f\|^2 = \sum_1^\infty c_k^2 = 0$. $\Rightarrow \|f\| = 0$ $\Rightarrow f = 0$.

$(\Leftarrow)$ Assume $\{\phi_k\}$ not complete. Then $\exists$ $g \ni$ $\|g\|^2 > \sum_1^\infty c_k^2$ where $c_k = \langle g, \phi_k \rangle$. Let $f \in L$ be $\ni$ ~~for~~ $c_k = \langle f, \phi_k \rangle$, $\|f\|^2 = \sum_1^\infty c_k^2$ (Riesz-Fischer)

And $f - g$ is $\perp$ to all the ϕ_k. $\therefore$ $\|f\|^2 \neq \|g\|^2 \Rightarrow f - g \neq 0$. done! //