Consider $(L, \|\cdot\|_\infty)$ = continuous real-valued functions on $[a,b]$ with $\|\cdot\|_\infty$ = sup-norm

if $\phi_n \to 0$ then $\|\phi_n\|_\infty \le C < \infty$.

On the other hand, $\phi_n \to 0 \iff f(\phi_n) \to 0$ for each $f \in L^*$.

Take $f = \psi_{x_0}$ where $x_0 \in [a,b]$.

then

$$\underbrace{\psi_{x_0}(\phi_n)}_{=\phi_n(x_0)} \to \psi_{x_0}(0) = 0$$

i.e. $\phi_n(x_0) \to 0$.

$\therefore$ In $(L, \|\cdot\|_\infty)$ $\phi_n \to 0 \Rightarrow$ 1) $\|\phi_n\|_\infty$ bounded

2) ϕ_n converges pointwise.

Notice that this is completely different from what happens if we take $\|\phi\| = \sqrt{\int_a^b |\phi(x)|^2 dx}$. Since in $(L, \|\cdot\|_2)$ we have

$$\phi_k = \cos\left(k \frac{2\pi x}{b-a}\right) \to 0$$

but this $\phi_k \not\to 0$ in $(L, \|\cdot\|_\infty)$

We have defined the weak topology τ_w on L.

i.e. given (L, τ) this determines L^*, and using L^* we define τ_w a topology on L, $\tau_w \subseteq \tau$.

Recall the strong topology on L^*. It was built on a local base of $\vec{0}$ of the form

$$\{U_{A,\varepsilon}\} \text{ where } \varepsilon > 0 \text{ and } A \subset L \text{ is bounded}$$

$$U_{A,\varepsilon} := \{ f \in L^* \mid |f(x)| < \varepsilon \ \ \forall x \in A \}.$$

There are two ways to view the construction of the weak topology on L^*

View #1: "replace A bounded with A finite".

i.e. build a topology based on a local base of $\vec{0}$ of the form

$$\{U_{x_1 \ldots x_n; \varepsilon}\} \text{ where } \varepsilon > 0, \ x_1 \ldots x_n \in L$$

and $U_{x_1 \ldots x_n; \varepsilon} = \{ f \in L^* \mid |f(x_i)| < \varepsilon \quad 1 \le i \le n \}$

View 2: "mimic the definition of the weak topology on L, but instead of finitely many $g_i \in (L^*)^*$, require that the finite set $\{g_i\}_1^n$ only be the evaluation functions $g_i = \varphi_{x_i}$ some $x_i \in L$.

then
$$\begin{aligned} U_{g_1 \ldots g_n; \varepsilon} &= \{ f \in L^* \mid |g_i(f)| < \varepsilon \quad 1 \le i \le n \} \\ &= \{ f \in L^* \mid |\varphi_{x_i}(f)| < \varepsilon \quad 1 \le i \le n \} \\ &= \{ f \in L^* \mid |f(x_i)| < \varepsilon \} = U_{x_1 \ldots x_n; \varepsilon} \end{aligned}$$

In any case, we define a weak topology on L^*
call it (L^*, τ_w)

By construction, $\tau_w \leq b$ where b = strong topology on L^*.

<u>thm</u>: $\{f_n\} \subseteq L^*$. Then $f_n \to f \in L^*$ if and only if
for each $x_i \in L$, $f_n(x_i) \to f(x_i)$

<u>proof</u>: Mimic the proof from before. WLOG assume $f_n \to \vec{0}$.
($\Rightarrow$) assume $f_n \to \vec{0}$. Fix $x_i \in L$. Then

$$U_{x_i, \varepsilon} = \{ f \in L^* \mid |f(x_i)| < \varepsilon \} \in \tau_w.$$

Since $f_n \to 0$, $\exists N$ so that $n \geq N$ then $f_n \in U_{x_i, \varepsilon}$.

$\Rightarrow$ $|f_n(x_i)| < \varepsilon$ $\Rightarrow$ $f_n(x_i) \to 0$ as desired.

($\Leftarrow$) Assume $f_n(x_i) \to 0$ for each $x_i \in L$. Let
$V \in \tau_w$ be a nbd of $\vec{0}$. Then $\exists$ $x_1 \ldots x_r$ and $\varepsilon > 0$
$\ni$ $U_{x_1 \ldots x_r; \varepsilon} \subseteq V$.

Since $f_n(x_1) \to 0$ $\exists N_1 \ni n \geq N_1 \Rightarrow |f_n(x_1)| < \varepsilon$

$f_n(x_r) \to 0$ $\exists N_r \ni n \geq N_r \Rightarrow |f_n(x_r)| < \varepsilon$

let $N = \max\{N_1, N_2, \ldots N_r\}$ then $n \geq N \Rightarrow f_n \in U_{x_1 \ldots x_r; \varepsilon} \subseteq V$
and done! //

In fact, most of the previous theorems go through, with care...

<u>thm</u>: Assume $\{f_n\} \subseteq L^*$ and $f_n \to f \in L^*$.
Assume $(L, \|\cdot\|)$ is complete (a Banach space)
Then $\exists C < \infty$ so that $\|f_n\| \le C$ $\forall n$.

!!! Why do we suddenly have to assume L is complete?
We didn't need that assumption before!!!

<u>proof</u>: <u>Note</u>: if we can show $f_n \to 0 \Rightarrow \|f_n\| \le C < \infty$ some C then we're done since

$$f_n \to f_0 \Rightarrow f_n - f_0 \to 0 \Rightarrow \|f_n - f_0\| \le C < \infty$$
$$\Rightarrow \|f_n\| \le \|f_0\| + C < \infty$$

done!

Assume $f_n \to 0$ but $\|f_n\| \not\le C < \infty$. Then given any closed ball $S[x_0, \varepsilon]$ in L, $\exists x \in S[x_0, \varepsilon]$ so that $\{|f_n(x)|\}$ is unbounded. Why? Assume not. Assume for each $x \in S[x_0, \varepsilon]$, $\{|f_n(x)|\}$ is bounded.

$$x \in S[x_0, \varepsilon] \iff x = x_0 + \varepsilon y \quad \text{some } y \in S[0,1]$$

$$\Rightarrow \varepsilon f_n(y) = f_n(x - x_0) = f_n(x) - f_n(x_0)$$

$$\Rightarrow \varepsilon \sup_{\|y\| \le 1} |f_n(y)| = \sup_{x \in S[x_0, \varepsilon]} |f_n(x) - f_n(x_0)| \le |f_n(x_0)| + \sup_{S[x_0, \varepsilon]} |f_n(x)|$$

by assumption,

$$\sup_{x \in S[x_0,\varepsilon]} |f_n(x)| < C < \infty$$

$$\Rightarrow \quad \varepsilon \sup_{\|y\| \leq 1} |f_n(y)| < C < \infty \quad \Rightarrow \quad \|f_n\| \leq \frac{C}{\varepsilon} \text{ for each } n.$$

$\times$ $\Rightarrow$ for each $S[x_0,\varepsilon]$,

$$\sup_{x \in S[x_0,\varepsilon]} |f_n(x)| = \infty.$$

$\Rightarrow$ given S_0 $\exists$ $x_1 \in S_0$ and $n_1 \ni |f_{n_1}(x_1)| > 1$.

Since f_{n_1} is cts on $(L, \|\cdot\|)$ $\exists$ closed ball $S_1 \subseteq S_0$ with $|f_{n_1}(x)| > 1$ $\forall x \in S_1$. Proceeding as before, we construct a subsequence $\{f_{n_k}\} \subset \{f_n\}$ and a nested family of closed balls in L with radii $\downarrow 0$. $\Rightarrow$ $x_\infty \in \bigcap_1^\infty S_n$!!! Need L complete !!!

by construction $|f_{n_k}(x_\infty)| > k$. But $f_n(x_\infty) \to 0$ by the weak convergence of $f_n \to 0$ $\times$

$\Rightarrow$ done. //

Corr: Let $(L, \|\cdot\|)$ be a Banach space.
Consider $\{f_n\} \subseteq L^*$. If $\{|f_n(x_0)|\}$ is bounded in $\mathbb{R}$ for each $x_0 \in L$ then $\{f_n\}$ is bounded in $(L^*, \|\cdot\|)$

Corr: Let $M \subseteq (L^*, \tau_w)$ be (weakly) bounded, (i.e. bounded in τ_w). If $(L, \|\cdot\|)$ is a Banach Space then M is (strongly bounded).

thm: Let $\{f_n\} \subset L^*$ where $(L, \|\cdot\|)$ is a Banach space then $f_n \to f \in L^*$ if $f_n(x) \to f(x)$ for every $x \in \Delta$ where Δ is any set whose linear hull is dense in $(L, \|\cdot\|)$.

proof: see K+F.

Example of $f_n \to f$

Let L = continuous functions on $[a,b]$
with $\|\cdot\| = \|\cdot\|_\infty$ = sup norm

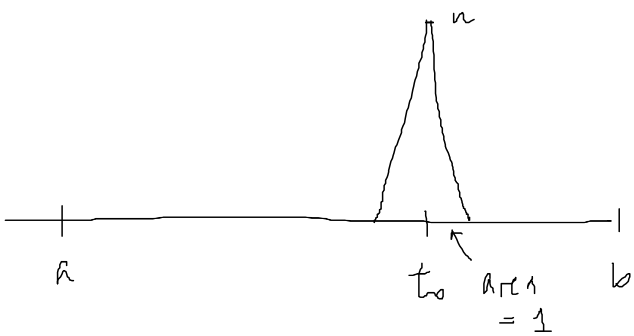
consider $\delta_{t_0}(x) = x(t_0)$ i.e. evaluate x at $t_0 \in [a,b]$.

This is the δ function. Claim: $\exists f_n \in L^*$ where f_n is "nicer" and $f_n \to \delta_{t_0}$.

First, define $\phi_n \in C([a,b])$ as follows:

$$\phi_n(t) = \begin{cases} 0 & a \le t < t_0 - 1/n \\ n^2(t-t_0)+n & t_0 - \frac{1}{n} \le t \le t_0 \\ -n^2(t-t_0)+n & t_0 < t \le t_0 + 1/n \\ 0 & t_0 - 1/n < t \end{cases}$$

for $n > \frac{2}{b-a}$



i.e. $\phi_n \geq 0$ on $[a,b]$

$\int_a^b \phi_n(t)\,dt = 1$

$\phi_n \in C([a,b])$

given $\phi_n \in C([a,b])$, I use them to define $f_n \in L^*$ as follows:

$$f_n(x) := \int_a^b x(t)\,\phi_n(t)\,dt$$

f_n is definitely linear. Is it continuous?
Need to check if it's bounded. i.e.

$$|f_n(x)| \le c_n \|x\|_\infty \quad \forall x \in L$$
some $c_n < \infty$

true: take $C = \int_a^b \phi_n(t)\,dt$

We have $\{f_n\} \subset L^*$.

Now claim $f_n \to \delta_{t_0}$. to show this, it suffices to show that if $x \in L$ then

$$f_n(x) \to \delta_{t_0}(x).$$

i.e. $\lim_{n\to\infty} \int_a^b x(t)\,\phi_n(t)\,dt = \delta_{t_0}(x) = x(t_0)$

i.e. $\lim_{n\to\infty} \int_{t_0-1/n}^{t_0+1/n} \phi_n(t)\,x(t)\,dt = x(t_0)$.

Recall MVT for integrals:

Let $w(t) \geq 0$ be integrable on $[A, B]$ and let $f(t)$ be continuous on. Then $\exists\, c \in [A, B]$ such that

$$\int_A^B f(t)\,w(t)\,dt = f(c)\int_A^B w(t)\,dt$$

$$\Rightarrow \int_{t_0-1/n}^{t_0+1/n} \phi_n(t)\,x(t)\,dt = x(t_n)$$ some $t_n \in [t_0 - 1/n, t_0 + 1/n]$

since $\int \phi_n(t) = 1$

$\phi_n \geq 0$

$\therefore \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x(t_n)$ where $t_n \in [t_0 - 1/n, t_0 + 1/n]$

$= x(t_0)$

since x continuous

$= \delta_{t_0}(x)$ as desired!

$\Rightarrow f_n \to \delta_{t_0}$

If we abuse notation and write

$$\delta_{t_0}(x) = \int_a^b x(t)\,\delta_{t_0}(t)\,dt$$

then we've shown that the "generalized function" δ_{t_0} is the weak limit of a sequence of continuous functions ϕ_n.

Note: Just needed $\int \phi_n = 1$, $\phi_n \geq 0$, and support(ϕ_n) collapsing to t_0

Note: what if x hadn't been continuous?

Note: didn't need L complete

$$\text{support}(\phi_n) = \overline{\{x \mid \phi_n(x) > 0\}}$$