Weak Topology and Weak Convergence.

Recall the HW problem where we considered

$X = \{$ continuous real valued functions on $[a,b]\}$

and for each $f \in X$ we defined a linear functional by $F_f(g) = \int_c^b g(x) f(x)\,dx$ $F_f : X \to \mathbb{R}$.

Clearly F_f is a linear functional. We then put a topology on X that made each F_f into a continuous linear functional. i.e. we chose

τ so that $F_f \in X^*$ if we look at the topological vector space (X, τ).

The topology was called the weak topology and it contains fewer open sets than the topology induced on X by $\langle f, g \rangle = \int_c^b f(x) g(x)\,dx$.

E.g. consider $\{\phi_k\} \subseteq X$ $\phi_k(x) = \cos\left(k \frac{2\pi x}{b-a}\right)$

then $\phi_k \rightharpoonup 0$ in (X, τ) ← weak topology

but $\phi_k \not\to 0$ in $(X, \langle \cdot \rangle)$

note: "$\rightharpoonup$" is the standard notation for weak convergence.

We want to define the weak topology for general topological vector spaces.

Let (L, τ) be a topological vector space. Let L^* be the dual space of (L, τ). We use L^* to define a new topology on L (Note: we needed some topology on L in order to define L^* in the first place. Or we could have just declared a set of linear functionals from the beginning and then asked for a topology on L that would make them continuous. Either way, we need the following.)

Given any $\varepsilon > 0$ and any finite collection $f_1 \ldots f_n \in L^*$. let

$$U_{f_1 \ldots f_n; \varepsilon} = \{ x \in L \mid |f_1(x)| < \varepsilon, |f_2(x)| < \varepsilon, \ldots |f_n(x)| < \varepsilon \}.$$

$\vec{0} \in U_{f_1 \ldots f_n; \varepsilon}$. Now let $\varepsilon > 0$ vary and let the finite set $\{f_1 \ldots f_n\} \subset L^*$ vary. Then $\{ U_{f_1 \ldots f_n; \varepsilon} \}$ is a local base at $\vec{0}$ and it will generate a topology on L. Further, the topology respects $+$ and $\cdot$ and (L, τ_w) is a topological vector space. We call τ_w the weak topology on L.

Clearly τ_w is the weakest topology that makes the linear functionals in L^* continuous.

Given (L, τ) original topology

(L, τ_w) weak topology induced by τ

We have (strongly) bounded sets in L and (weakly) bounded sets. i.e. $A \subset L$ is <u>strongly bounded</u> if $\exists$ $U \in \tau$, $\vec{0} \in U$ so that

$$A \subseteq \alpha U \quad \text{some } \alpha \in K.$$

A is <u>weakly bounded</u> if $\exists$ $U \in \tau_w$, $\vec{0} \in U$ so that

$$A \subseteq \alpha U \quad \text{some } \alpha \in K.$$

Similarly we have strongly convergent sequences and weakly convergent sequences. We expect it is harder to converge strongly since τ is larger than τ_w.

Thm: Consider $\{x_n\} \subseteq (L, \tau)$

Then x_n is weakly convergent to $x_0 \in L$ if and only if $f(x_n) \to f(x)$ for each $f \in L^*$.

proof: WLOG, assume $x_n \to \vec{0}$.

($\Leftarrow$) Assume $f(x_n) \to f(\vec{0}) = 0$ for each $f \in L^*$.

Let $U \in \tau_w$ be a neighborhood of $\vec{0}$. Then $\exists\, f_1 \dots f_n$ and $\varepsilon > 0$ so that

$$U_{f_1 \dots f_n; \varepsilon} \subseteq U.$$

Since $f_1(x_m) \to 0$ $\exists\, N_1$ so that $|f_1(x_m)| < \varepsilon$ $\forall m \geq N_1$

$\vdots$

$f_n(x_m) \to 0$ $\exists\, N_n$ so that $|f_n(x_m)| < \varepsilon$ $\forall m \geq N_n$

Let $N = \max\{N_1 \dots N_n\}$ Then if $m \geq N$ we have $x_m \in U_{f_1 \dots f_n; \varepsilon} \subseteq U$. $\Rightarrow$ $x_m \to \vec{0}$ as desired.

($\Longrightarrow$) assume $x_n \to \vec{0}$. Let $f \in L^*$. Fix $\varepsilon > 0$.

Then $U_{f;\varepsilon}$ is in τ_w. $\Rightarrow \exists\, N$ so that if $n \geq N$ then $x_n \in U_{f;\varepsilon}$. $\Rightarrow$ $n \geq N \Rightarrow |f(x_n)| < \varepsilon$ and done! //

Recall boundedness. From before, we know that if (L, τ) is a topological vector space and $x_n \to \vec{0}$ then $\{x_n\}$ is a bounded set. (This involved showing that every open set containing $\vec{0}$ contains an absorbing open subset containing $\vec{0}$.

Specifically if $x_n \rightharpoonup \vec{0}$ (i.e. $x_n \to 0$ in (L, τ_w)) then $\{x_n\}$ is bounded in (L, τ_w). If (L, τ) comes from a norm, however then $\{x_n\}$ is actually bounded in (L, τ)!

i.e. $x_n \rightharpoonup 0$ then $\{x_n\}$ is strongly bounded

Theorem. Consider $(L, \|\cdot\|)$. Assume $x_n \rightharpoonup x_0 \in L$ (x_n converges weakly to x_0). Then $\{x_n\}$ is strongly bounded. i.e. $\exists\, C < \infty$ so that

$$\|x_n\| \leq C \qquad \forall n.$$

proof: Assume $x_n \rightharpoonup x_0$ but $\|x_n\|$ is not bounded. First, without loss of generality, we can assume $x_n \rightharpoonup 0$ since we can consider

$$x_n - x_0 \rightharpoonup 0$$ and just show $\|x_n - x_0\|$ is bounded.

So $x_n \to 0$ but $\|x_n\|_L$ is not bounded (by assump.)
This implies that if $f_0 \in L^*$ and $\varepsilon > 0$ then the set

$$M_{f_0,\varepsilon} = \{ |f(x_n)| \mid f \in S[f_0,\varepsilon],\ n \in \mathbb{N} \}$$

is an unbounded set in $\mathbb{R}$. (Why? we'll explain in a moment... let's just continue for a bit...).

So, let S_0 be a closed sphere in L^*. Then $\exists$ $g_0 \in S_0$ and $x_{n_1} \in \{x_n\}$ so that

$$|g_0(x_{n_1})| > 1.$$

(since we've assumed $M_{f_0,\varepsilon}$ is unbounded $\forall f_0, \forall \varepsilon$).
Therefore, if $\tilde{g}$ is close to g_0, we'll have $|\tilde{g}(x_{n_1})| > 1$ too. Let S_1 be a closed sphere contained in S_0 so that $|g(x_{n_1})| > 1$ if $g \in S_1$.

Now, $\exists$ $g_1 \in S_1$ and $x_{n_2} \in \{x_n\}$ so that $|g_1(x_{n_2})| > 2$.
By the same argument as above, $\exists$ closed sphere $S_2 \subset S_1 \subset S_0$ so that $|g(x_{n_2})| > 2$ if $g \in S_2$
We continue in this way, constructing a nested sequence of closed spheres $\{S_k\}$ and a

subsequence $\{x_{n_k}\}$ of $\{x_n\}$ so that

$$|g(x_{n_i})| > i \quad \text{for all } g \in S_k$$

Also, we can take the radii of the spheres to be going to zero. Now, since L^* is <u>complete</u>, $\exists\ \tilde{f} \in L^*$ so that $\tilde{f} \in \bigcap_{1}^{\infty} S_k$. And by construction, $|\tilde{f}(x_{n_k})| \to \infty$ as $k \to \infty$. This is a contradiction since $x_n \to 0 \Rightarrow \tilde{f}(x_n) \to 0$ $\forall \tilde{f} \in L^*$. $\Rightarrow$ our assumption that $\|x_n\|_L$ is unbounded in $\mathbb{R}$ must be false!

This finishes the proof, once we've proven that $\|x_n\|_L$ not bounded $\Rightarrow M_{f_0,\varepsilon}$ is an unbounded set in $\mathbb{R}$ for any $f_0 \in L^*$, any $\varepsilon > 0$. We'll do this by contradiction. i.e. we'll show that if $\exists f_0 \in L^*$ and $\varepsilon > 0$ so that $M_{f_0,\varepsilon}$ is bounded in $\mathbb{R}$, then $\|x_n\|_L$ is bounded.

Assume $f_0 \in L^*$, $\varepsilon > 0$ is such that $M_{f_0,\varepsilon}$ is bounded.

Then $|f(x_n)| \le c < \infty$ for all n and for all f so that $\|f - f_0\|_{L^*} < \varepsilon$. i.e.

if $f \in L^*$ satisfies $\|f - f_0\|_{L^*} < \varepsilon$ then $f = f_0 + \varepsilon g$ for some $g \in L^*$ with $\|g\|_{L^*} \le 1$. i.e. if $g \in L^*$ and $\|g\|_{L^*} \le 1$ then $g = \frac{1}{\varepsilon}(f - f_0)$ for some $f \in L^*$, $\|f - f_0\|_{L^*} < \varepsilon$.

$$\Rightarrow \quad |g(x_n)| = \frac{1}{\varepsilon}|f(x_n) - f_0(x_n)| \le \frac{1}{\varepsilon}\left(|f(x_n)| + |f_0(x_n)|\right)$$

$$\le \frac{2c}{\varepsilon}$$

since $M_{f_0, \varepsilon}$ is a bounded set in $\mathbb{R}$.

$$\Rightarrow \sup_{\substack{g \in L^*, \\ \|g\|_{L^*} \le 1}} |g(x_n)| \le \frac{2c}{\varepsilon}$$

$$\Rightarrow \sup_{\substack{g \in L^*, \\ \|g\|_{L^*} \le 1}} |\Psi_{x_n}(g)| \le \frac{2c}{\varepsilon}$$

$\Rightarrow \|\Psi_{x_n}\|_{L^{**}} \le \frac{2c}{\varepsilon}$. Since the natural mapping

$\Pi : L \to L^{**}$ with $\Pi(x) = \Psi(x)$ is an isometry, we know $\|x_n\|_L = \|\psi_{x_n}\|_{L^{**}}$.

$\Rightarrow$ we've just shown

$$\|x_n\|_L \leq \frac{2C}{\varepsilon}.$$

and since the right hand side is indep of n, we've found a uniform upper bound for $\|x_n\|_L$, as desired!

this finishes the proof. //

Corr: Let $\{x_n\} \subseteq L$, where L is a normed top. vector space, be such that

$$\{|\psi_{x_n}(f)|\} \text{ is bounded for each } f \in L^*$$

Then $\{x_n\}$ is strongly bounded. (i.e. $\|x_n\|_L \leq C_1 < \infty$ some C_1.)

Corr: Let $M \subseteq L$ be weakly bounded, where L is a normed top. vector space. Then M is strongly bounded (i.e. $M \subseteq S[0,R]$ for some $R < \infty$.)

proof: Assume not. i.e. M is not strongly bounded. Then $\exists \{x_n\} \subseteq M$ so that $\|x_n\|_L \to \infty$. Since M is weakly bounded, if $f \in L^*$ then M is absorbed by $U_{f,\varepsilon}$ i.e. $M \subseteq \alpha U_{f,\varepsilon}$ for some α. i.e. $|f(x)| < \alpha\varepsilon$ for all $x \in M$

Specifically,

$$|f(x_n)| < \alpha\varepsilon \quad \forall\ x_n \in \{x_n\}$$

This shows that $\{|\Psi_{x_n}(f)|\}$ is bounded for each $f \in L^*$. $\Rightarrow$ by previous corollary, $\|x_n\|$ is bounded. contradicting $\|x_n\| \to \infty$.