Notation: $(L, \| \|) \leftrightarrow (L', ||| \; |||)$

if $\exists$ isometry $\pi : L \to L'$ so that $\pi(L) = L'$,

i.e. $|||\pi(x)||| = \|x\| \quad \forall x \in L$.

Then $((\ell^p)^*, \|\cdot\|) \leftrightarrow (\ell^q, \| \|_q)$ if $\frac{1}{p} + \frac{1}{q} = 1 \quad 1 < p < \infty$

where $\pi : \ell^q \to (\ell^p)^*$ is defined by

$$\pi(f)(g) = \sum_1^\infty f_k g_k \quad \text{for each } g \in \ell^p.$$

Similarly, if $c_0 \subseteq \ell^\infty$ is the set of bounded sequences such that $x_k \to 0$ as $k \to \infty$ then

$$((c_0)^*, \|\cdot\|) \leftrightarrow (\ell^1, \|\cdot\|_1)$$

where $\pi : \ell^1 \to (c_0)^*$ is defined by

$$\pi(f)(x) = \sum_1^\infty f_k x_k \quad \text{for each } x \in c_0$$

Note: when we have $(L, \|\cdot\|)$ inducing a norm on L^*, we call $(L^*, \| \|)$ the strong topology

and $((\ell^1)^*, \|\cdot\|) \leftrightarrow (\ell^\infty, \|\cdot\|_\infty)$

where $\pi : \ell^\infty \to (\ell^1)^*$ is defined by

$$\pi(f)(x) = \sum_1^\infty f_k x_k \quad \text{for each } x \in \ell^1.$$

thm: Let $(L, \langle \rangle)$ be a Hilbert space. Let $\|\cdot\|_{\langle\cdot\rangle}$ be the norm on L given by $\|x\|_{\langle\rangle} = \sqrt{\langle x, x\rangle}$

then $(L^*, \| \|) \leftrightarrow (L, \|\cdot\|_{\langle\rangle})$

proof: first, define $\pi : L \to L^*$. Let $a \in L$. then define $f_a : L \to \mathbb{R}$ by $f_a(x) = \langle x, a\rangle$.

f_a is certainly a linear functional. Is it bounded?

$$|f_a(x)| = |\langle x, a\rangle| \le \|a\|_{\langle\cdot\rangle} \|x\|_{\langle\cdot\rangle} \quad \forall x \in L$$

$\Rightarrow f_a \in L^*$. Moreover $\|f_a\| \le \|a\|_{\langle\cdot\rangle}$. Since $|f_a(a)| = \|a\|^2_{\langle\cdot\rangle}$, we know $\|f_a\| \ge \|a\|_{\langle\cdot\rangle}$.

$\Rightarrow \|f_a\| = \|a\|_{\langle\cdot\rangle}$. Since $\pi(a) = f_a$, we've found $\pi : L \to L^*$ s. that $\|\pi(a)\| = \|a\|_{\langle\cdot\rangle}$ $\forall a \in L$.

this means all we need to do is show $\pi(L) = L^*$.

i.e. given $f \in L^*$ find $a \in L \ni f(x) = \langle x, a\rangle$ $\forall x \in L$.

we proved such an a exists last week done! //

<u>thm:</u> $((\ell^p)^*, \|\cdot\|) \leftrightarrow (\ell^q, \|\cdot\|_q)$ if $\frac{1}{p} + \frac{1}{q} = 1$ $\quad 1 < p < \infty$

<u>fact:</u> If we're given $(L, \|\cdot\|)$, the Hahn Banach theorem guarantees nontrivial linear functionals.

<u>why?</u> fix $x_0 \in L$, $\|x_0\| = 1$. Let $L_0 = \text{span}\{x_0\}$. We define a linear functional on L_0 by

$$f_0(\alpha x_0) = \alpha$$

this is definitely linear. Also, $|f_0(\alpha x_0)| = |\alpha| = \|\alpha x_0\|$

$\Rightarrow \|f_0\| = 1$ and we've found a convex functional defined on $L \ni |f_0(x)| \le p(x)$ $\forall x \in L_0$. (i.e. $p(x) = \|x\|$)

By the Hahn-Banach theorem, we can extend f_0 from L_0 to all of L in such a way that f is linear and $|f(x)| \le p(x)$ $\forall x \in L$, i.e. $|f(x)| \le \|x\|$ $\forall x \in L$

Thus we can use our elements of L to construct linear functionals on L.

Q: Given (L, τ) you can define the vector space of continuous linear functionals on L. Call it L^*.
How do we give L^* a topology when we don't have a norm on L?

A. Use a base at $\vec{0}$ built on the bounded sets in L.

Defn: Let (L, τ) be a topological vector space and let L^* be the vector space of continuous linear functionals on L. Then the strong topology on L^* is the topology generated by the neighborhood base at 0:

$$U_{A,\varepsilon} = \{ f \in L^* \mid |f(x)| < \varepsilon \quad \forall x \in A \}$$

where $\varepsilon > 0$ and $A \subset L$ is a bounded set.

K+F denotes L^* w/ strong topology by (L, b).

Okay, given

$(L, \|\,\|)$ we have $(L^*, \|\cdot\|)$ the dual space w/ the strong topology.

Similarly, given (L, τ) we have (L^*, b)

In both cases, $(L^*, \|\,\|)$ and (L^*, b) are vector spaces that have dual spaces of their own.

$((L^*)^*, \|\cdot\|)$ is the dual space of L^* with the norm induced by the norm on L^*. And we have $((L^*)^*, b^*)$ the dual space of L^* w/ strong topology induced by the bounded sets in L^*.

<u>thm</u>: given (L, τ) w/ dual space (L^*, b). Let $x_0 \in L$ be fixed. Then $\Psi_{x_0} : L^* \to \mathbb{R}$ defined by $\Psi_{x_0}(f) = f(x)$ is a continuous linear functional on L^*. (i.e. $\Psi_{x_0} \in L^{**}$).

<u>proof</u>: $\Psi_{x_0}(\alpha f + \beta g) = (\alpha f + \beta g)(x_0)$

$= \alpha f(x_0) + \beta g(x_0) = \alpha \Psi_{x_0}(f) + \beta \Psi_{x_0}(g)$.

$\Rightarrow \Psi_{x_0}$ is linear fxnl. Is it continuous?

We have $\psi_{x_0} : L^* \to \mathbb{R}$. It's linear so it suffices to check that ψ_{x_0} is continuous at 0. i.e. If V is an open set containing $\psi_{x_0}(0)$ then $\psi_{x_0}^{-1}(V)$ is an open set. Note: $\psi_{x_0}(0) = 0$.

Let $f \in \psi_{x_0}^{-1}(V)$. Then $\psi_{x_0}(f) \in V$ and $\exists\, \varepsilon > 0$ s.t. that

$\forall \lambda \in \mathbb{R}$, $|\lambda - \psi_{x_0}(f)| < \varepsilon \Rightarrow \lambda \in V$. i.e

$S((\psi_{x_0}(f), \varepsilon)) \subseteq V$. I claim that $\psi_{x_0}^{-1}(S((\psi_{x_0}(f), \varepsilon)))$ is an open set containing f. Clearly, $\psi_{x_0}^{-1}(S((\psi_{x_0}(f), \varepsilon))) \subseteq \psi_{x_0}^{-1}(V)$ so if I can show ~~it~~ $\psi_{x_0}^{-1}(S((\psi_{x_0}(f), \varepsilon)))$ is open, then I'm done.

$$g \in \psi_{x_0}^{-1}(S((\psi_{x_0}(f), \varepsilon))) \iff \psi_{x_0}(g) \in S((\psi_{x_0}(f), \varepsilon))$$
$$\iff |\psi_{x_0}(g) - \psi_{x_0}(f)| < \varepsilon$$
$$\iff |g(x_0) - f(x_0)| < \varepsilon.$$

Since $\{x_0\}$ is a bounded set, $U_{\{x_0\},\varepsilon}$ is an open nbd of 0 in L^*.
$\Rightarrow f + U_{\{x_0\}\varepsilon}$ is an open nbd of f in L.

And $g \in f + U_{\{x_0\}\varepsilon} \iff g = f + h$ for some $h \in L^*$ $\ni |h(x_0)| < \varepsilon$.

well, $h = g - f$ and so $g \in f + U_{\{x_0\}\varepsilon} \iff |g(x_0) - f(x_0)| < \varepsilon$.

This shows that

$$\psi_{x_0}^{-1}(S((\psi_{x_0}(f), \varepsilon))) = f + U_{\{x_0\}\varepsilon} \Rightarrow \psi_{x_0} \text{ is continuous}$$

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We call $\pi : L \to L^{**}$

defined by $\pi(x_0) = \psi_{x_0}$ the natural mapping of L into L^{**}. Clearly, $\pi(L) \subseteq L^{**}$ and

π is a linear mapping. If L^* has sufficiently many linear functionals (i.e. given $x_1 \neq x_2$ in L then $\exists f \in L^*$ with $f(x_1) \neq f(x_2)$) then $\pi : L \to L^{**}$ is 1:1.

If $\pi(L) = L^{**}$ we call (L, τ) <u>semireflexive</u>.

If $\pi(L) = L^{**}$ and π and π^{-1} are continuous then we call L <u>reflexive</u>. In this case we use the notation $(f, x) = f(x)$

$$\underset{\in L^*}{f}, \quad \underset{\in L}{x}$$

<u>thm.</u> If $(L, \|\cdot\|)$ then $\pi : L \to L^{**}$ is an isometry.

<u>corr:</u> If $(L, \|\cdot\|)$ and $\pi(L) = L^{**}$ then $(L, \|\cdot\|)$ is reflexive.

<u>proof:</u> π an isometry $\Rightarrow$ π and π^{-1} are cts.

proof of theorem.

we want to show $\|\pi(x_0)\| = \|x_0\|$ $\forall x_0 \in L$.

first recall

$$\|\pi(x_0)\| = \sup_{\substack{f\neq 0\\ \|f\|\le 1}} \frac{|\pi(x_0)(f)|}{\|f\|} \qquad f\in L^*$$

$$= \sup_{\substack{\|f\|\le 1\\ f\neq 0}} \frac{|f(x_0)|}{\|f\|}$$

and $|f(x_0)| \le \|f\|\cdot\|x_0\| \Rightarrow \|\pi(x_0)\| \le \sup_{\substack{\|f\|\le 1\\ f\neq 0}} \frac{\|f\|\cdot\|x_0\|}{\|f\|}$

$\Rightarrow \|\pi(x_0)\| \le \|x_0\|$.

Now, I want to find $f\in L^* \ni |\pi(x_0)f| = \|f\|\|x_0\|$
this would then imply $\|\pi(x_0)\| \ge \|x_0\|$ and thus
$\|\pi(x)\| = \|x\|$ as desired.

Since x_0 is fixed, we can define $L_0 = \text{span}\{x_0\}$ and
define f on L_0 by $f(\lambda x_0) = \lambda$. then assume c_1 satisfies

$$|\lambda| = |f(\lambda x_0)| \le c_1\|\lambda x_0\| = c_1|\lambda|\,\|x_0\|$$

$\Rightarrow$ $1 \le c_1\|x_0\| \Rightarrow \frac{1}{\|x_0\|} \le c_1$ taking inf over such c_1. this
is $\|f\|$ on L. $\Rightarrow \|f\| = \frac{1}{\|x_0\|}$. (and we've used Hahn-Banach here
to extend f from L_0 to L w/o
increasing its norm.)

Okay... we constructed f in L^* so that $\|f\| = \frac{1}{\|x_0\|}$

Furthermore,

$$|\pi(x_0)(f)| = |f(x_0)| = 1 = \frac{1}{\|x_0\|} \cdot \|x_0\| = \|f\| \cdot \|x_0\|$$

$\Rightarrow \|\pi(x_0)\| \geq \|x_0\|$ as desired //