From last time, we looked at continuous (bounded) linear functionals on $(L, \|\cdot\|)$ and $(L, \langle\cdot\rangle)$ and (L, τ).

For $(L, \|\cdot\|)$, we were able to define the norm of a bounded linear functional:

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}$$

We know that the sum of linear fcnls is a linear functional. And if they're both bounded, then the sum is bounded. Similarly, if $\alpha \in \mathbb{R}$ (or $\mathbb{C}$) and f is a bounded linear functional, then αf is a bounded linear functional.

So the space of bounded linear functionals is a vector space, with a norm!

<u>Note:</u> even if L didn't have a norm, the space of linear functionals is a vectorspace. It's not immed. obvious how to put a topology on this space, is all.

Given $(L, \|\cdot\|)$, let L^* be the set of bounded linear functionals on L. Then L^* is a vector space. Further if $f \in L^*$ and we define

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}$$

then $\|\cdot\|$ on L^* is a <u>norm.</u> The corresponding topology on L^* is called the strong topology.

Note. I'm being cavalier with the notation "$\|\cdot\|$". The $\|\cdot\|$ on L is different from the norm on L^*. But the norm on L^* is induced by the norm on L.

e.g. if $L = \mathbb{R}^n$ and $\|x\| = \sqrt{\sum_1^n x_i^2}$

then L^* is isomorphic to $\mathbb{R}^n$ with the norm on L^* being $\sqrt{\sum_1^n x_i^2}$.

if $L = \mathbb{R}^n$ and $\|x\| = \sqrt[p]{\sum_1^n x_i^p}$, $1 < p < \infty$

then L^* is isomorphic to $\mathbb{R}^n$ with the norm on L^* being $\sqrt[q]{\sum_1^n x_i^q}$ where $\frac{1}{p} + \frac{1}{q} = 1$

Why? Let $L = \mathbb{R}^n$ and let $e_1 \ldots e_n$ be a basis of L. $\Rightarrow$ $x \in L$ has a unique representation as

$$x = \sum_1^n x_k e_k$$

Let $f \in L^*$. Then $f(x) = \sum_1^n f(e_k) x_k$

i.e. f is completely determined by what it does to $e_1 \ldots e_n$. i.e.

$$f \longleftrightarrow \begin{pmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{pmatrix} \in \mathbb{R}^n$$

Choosing the norm on L^* is something we'll do shortly.

thm: $(L^*, \|\cdot\|)$ is complete.

[Whoah!?! Don't we care whether L is complete? No. We'll only use that $\mathbb{R}$ (or $\mathbb{C}$) is complete.]

proof: Let $\{f_n\}$ be a Cauchy sequence of functionals in L^*. We want to show $f_n \to f$ in L^*.

Since $\{f_n\}$ is Cauchy, given $\varepsilon > 0$ $\exists N$ so that $m, n \geq N \Rightarrow \|f_n - f_m\| < \varepsilon$

Fix $x \in L$. Then

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\| \, \|x\| < \varepsilon \|x\|.$$

$\Rightarrow \{f_n(x)\}$ is Cauchy in $\mathbb{R}$. $\Rightarrow \lim_{n\to\infty} f_n(x) = f(x)$ exists. So for each $x \in L$, I can define $f(x) \in \mathbb{R}$.

Q1: is f a linear functional?

Q2: is f a bounded linear functional?

Q3: is f the limit of f_n? i.e. $\|f_n - f\| \to 0$. (we've defined f via pointwise limits above...)

$$f(\alpha x + \beta y) = \lim_{n\to\infty} f_n(\alpha x + \beta y) = \lim_{n\to\infty} \alpha f_n(x) + \beta f_n(y) = \alpha f(x) + \beta f(y).$$

So f is linear.

choose $N \ni \quad \|f_n - f_m\| \leq 1 \quad \forall n, m \geq N$.

then $\|f_n - f_N\| \leq 1 \quad \forall\, n \geq N$

$$\Rightarrow \quad \|f_n\| = \|f_n - f_N + f_N\| \leq 1 + \|f_N\| \qquad \forall n \geq N.$$

$$\Rightarrow \quad |f_n(x)| \leq (1 + \|f_N\|)\, \|x\| \qquad \forall x \in L.$$

$$\Rightarrow \quad \lim_{n \to \infty} |f_n(x)| \leq (1 + \|f_N\|)\, \|x\|.$$

$$\Rightarrow \quad |f(x)| \leq (1 + \|f_N\|)\, \|x\|.$$

$\Rightarrow$ f is a bounded linear functional. (Note! we didn't show what the norm of f equals, we just showed that whatever it is, it's finite. Hence f is a bounded linear functional.)

Finally, we show $\|f_n - f\| \to 0$ as $n \to \infty$. Fix $\varepsilon > 0$.
We want to show $\exists\, N_\varepsilon \ni \quad \|f_n - f\| < \varepsilon \quad \forall\, n \geq N_\varepsilon$.

First, choose $N \ni \quad \|f_m - f_N\| < \varepsilon/3 \quad \forall\, m \geq N$.

For this ε and this N, $\exists\, x_{N\varepsilon} \ni$

$$\|f_N - f\| \leq \frac{|f_N(x_{N\varepsilon}) - f(x_{N\varepsilon})|}{\|x_{N\varepsilon}\|} + \frac{\varepsilon}{3} = |f_N(u_{N\varepsilon}) - f(u_{N\varepsilon})| + \varepsilon/3$$

for $u_{N\varepsilon} = \dfrac{x_{N\varepsilon}}{\|x_{N\varepsilon}\|}$.

$$\Rightarrow \|f_N - f\| \le |f_N(u_{N\varepsilon}) - f_m(u_{N\varepsilon})| + |f_m(u_{N\varepsilon}) - f(u_{N\varepsilon})| + \varepsilon/3$$

$$\le \|f_N - f_m\| \, \|u_{N\varepsilon}\| + |f_m(u_{N\varepsilon}) - f(u_{N\varepsilon})| + \varepsilon/3$$

$$< \frac{2\varepsilon}{3} + |f_m(u_{N\varepsilon}) - f(u_{N\varepsilon})| \quad \text{if } m \ge N.$$

Now, the LHS is independent of m. On the right-hand side, we can take $m \to \infty$.

$\Rightarrow$ for m large enough, $|f_m(u_{N\varepsilon}) - f(u_{N\varepsilon})| < \varepsilon/3$

and $\|f_N - f\| < \varepsilon \Rightarrow f_n \to f$! //

thm 2: Let $(L, \langle\,\rangle)$ be a Hilbert Space. Then, given $x_0 \in L$, $x \to \langle x, x_0 \rangle$ is a bounded linear functional on L. Further, given $f \in L^*$, $\exists\, x_0 \in L$ so that $f(x) = \langle x, x_0 \rangle \ \forall\, x \in L$.

proof: K+F

corr: The correspondence $x_0 \Leftrightarrow f$ is an isomorphism between $(L, \|\,\|)$ and $(L^*, \|\,\|)$ (if $(L, \langle\,\rangle)$ is a Hilbert space ex: let $(L, \| \|)$ be $\ell^p(\mathbb{R}, \mathbb{N})$ where $p \in (1, \infty)$.

i.e. $x \in L$ if x is a sequence of real numbers such that $\sum_1^\infty |x_i|^p < \infty$

Claim: $(L^*, \| \|) = \ell^q(\mathbb{R}, \mathbb{N})$ where $\frac{1}{p} + \frac{1}{q} = 1$.

NOTE IT IS IMPORTANT THAT

$1 < p < \infty$

specifically, I want to show that

given $\tilde{f} \in L^*$ $\exists f \in \ell^q(\mathbb{R}, \mathbb{N})$

so that $\tilde{f} \longleftrightarrow f$

proof: let $f \in \ell^q(\mathbb{R}, \mathbb{N})$. Let $x \in \ell^p(\mathbb{R}, \mathbb{N})$.

then $x \longrightarrow \sum_1^\infty x_n f_n$ is a linear functional.

Further, it is a bounded linear functional since

$$|\tilde{f}(x)| = \left| \sum_1^\infty x_n f_n \right| \leq \sqrt[p]{\sum_1^\infty |x_n|^p} \sqrt[q]{\sum_1^\infty |f_n|^q}$$

$$= \|x\| \sqrt[q]{\sum_1^\infty |f_n|^q}$$

$$\Rightarrow \|\tilde{f}\| \leq \|f\|$$

norm on L^* $\quad$ ℓ^q norm.

This shows that $\|\tilde{f}\| \leq \|f\|_q$ Now I want to show $\|\tilde{f}\| \geq \|f\|_q$ I'll do this by constructing a sequence of vectors $x^n \in \ell^p$ so that $\lim_{n\to\infty} |\tilde{f}(x^n)| = \|f\|_q$ (this would show $\|\tilde{f}\| \geq \|f\|_q$.)

let $e_1 = (1, 0, \dots) \in \ell^p$

$e_2 = (0, 1, 0 \dots) \in \ell^p$

$\vdots$

$e_n = (0, 0 \dots 0, 1, 0 \dots) \in \ell^p$

$n^{\underline{th}}$ index

define

$$x^n = \frac{1}{(\|f\|_q)^{q/p}} \sum_1^n \frac{|f_k|^q}{f_k} e_k \qquad \left(\text{where we set } \frac{|f_k|^q}{f_k} = 0 \text{ if } f_k = 0\right).$$

First, $x^n \in \ell^p$ (trivially) and

$$\|x^n\|_p = \frac{1}{(\|f\|_q)^{q/p}} \sqrt[p]{\sum_1^n (|f_k|^{q-1})^p}$$

$$= \frac{1}{(\|f\|_q)^{q/p}} \sqrt[p]{\sum_1^n |f_k|^q} \leq \frac{1}{(\|f\|_q)^{q/p}} \cdot (\|f\|_q)^{q/p} = 1$$

$\Rightarrow \|x^n\|_p \leq 1$ for each n.

Further, $\left|\tilde{f}(x^n)\right| = \left|\sum_1^n |f_k|^q\right| \frac{1}{(\|f\|_q)^{q/p}}$

and $\lim_{n\to\infty} |\tilde{f}(x^n)| = (\|f\|_q)^q \frac{1}{(\|f\|_q)^{q/p}} = (\|f\|_q)^{q - \frac{q}{p}}$

$$= (\|f\|_q)^{q(1-\frac{1}{p})} = \|f\|_q .$$

$$\Rightarrow \quad \|\tilde{f}\| \geq \|f\|_q \quad \Rightarrow \quad \|\tilde{f}\| = \|f\|_q$$

this shows that given $f \in \ell^q$, we can define $\tilde{f} \in L^*$ ($\tilde{f} \in (\ell^p)^*$) so that $\|\tilde{f}\| = \|f\|_q$. We want to show $\ell^q \to (\ell^p)^*$ is 1:1 and onto.

1:1 assume $f, g \in \ell^q$ and

$$f \to \tilde{f} \in (\ell^p)^* \leftarrow g$$

i.e. $\tilde{f}(x) = \sum_1^\infty x_n f_n = \sum_1^\infty x_n g_n \quad \forall\, x \in \ell^p$

specifically, $\tilde{f}(e_n) = f_n = g_n \quad \forall\, k \Rightarrow f = g \Rightarrow$ 1:1

Onto: given $\tilde{f} \in (\ell^p)^*$, I want to find $f \in \ell^q$ so that $\tilde{f}(x) = \sum_1^\infty x_n f_n$

First, I define $f_n = \tilde{f}(e_n)$. This gives me a sequence of numbers. I want to show that

a) $\tilde{f}(x) = \sum_1^\infty x_n f_n \quad \forall\, x \in \ell^p$

b) $\{f_n\} \in \ell^q$

a). I know $\| x - \sum_1^n x_n e_n \|_p \to 0$ as $n \to \infty$

Since $\tilde{f}$ is cts, this implies

$$\left| \tilde{f}(x) - \tilde{f}\left(\sum_1^n x_n e_n \right) \right| \to 0$$

$$\left| \tilde{f}(x) - \sum_1^n x_n \tilde{f}(e_n) \right| \to 0$$

$$\left| \tilde{f}(x) - \sum_1^n x_n f_n \right| \to 0 \Rightarrow \tilde{f}(x) = \sum_1^\infty x_n f_n. \checkmark$$

Now I just want to show $\{f_n\} \in \ell^q$.

let $\quad x^n = \sum_1^n \frac{|\tilde{f}(e_n)|^q}{\tilde{f}(e_n)} e_n \qquad \left(\text{where } \frac{|\tilde{f}(e_n)|^q}{\tilde{f}(e_n)} = 0 \text{ if } \tilde{f}(e_n) = 0\right)$

$$\Rightarrow |\tilde{f}(x^n)| = \left|\sum_1^n |\tilde{f}(e_n)|^q\right|, \quad \sum_1^n |\tilde{f}(e_n)|^q \le \|\tilde{f}\| \, \|x^n\|_p$$

now $\quad \|x^n\|_p = \sqrt[p]{\sum_1^n |\tilde{f}(e_n)|^{(q-1)p}} = \sqrt[p]{\sum_1^n |\tilde{f}(e_n)|^q}$

$$\Rightarrow \sum_1^n |\tilde{f}(e_n)|^q \le \|\tilde{f}\| \sqrt[p]{\sum_1^n |\tilde{f}(e_n)|^q}$$

$$\Rightarrow \left(\sum_1^n |\tilde{f}(e_n)|^q\right)^{\frac{1}{q}} \le \|\tilde{f}\|.$$

recall, $\tilde{f}(e_n) =: f_n$.

$$\Rightarrow \left(\sum_1^n |f_n|^q\right)^{1/q} \le \|\tilde{f}\| \quad \forall n.$$

$$\Rightarrow \left(\sum_1^\infty |f_n|^q\right)^{1/q} \le \|\tilde{f}\|$$

and $\{f_n\} \in \ell^q$. ✓

this proves $(\ell^p)^* = \ell^q$ if $1 < p < q$. $\Rightarrow ((\ell^p)^*)^* = \ell^p$

Note: let $c_0 = \ell^\infty$ s.t. $\lim_{k\to\infty} x_k = 0$. then $(c_0)^* = \ell^1$.
and $(\ell^1)^* = \ell^\infty$. $\Rightarrow ((c_0)^*)^* \ne c_0$