Recall for topological vector spaces (L, τ)

thm 1: Let f be a linear functional on (L, τ) and suppose f is cts at some $x_0 \in L$. Then f is cts at every point of L

thm 2: Let f be a linear functional on (L, τ) Then f is continuous on L if and only if f is bounded on some neighborhood of $\vec{0}$.

thm 3: Let f be a linear functional on (L, τ)
1) if f is continuous on L then f is bounded on every bounded set in L.
2) if (L, τ) is first countable and f is bounded on every bounded set in L, then f is cont. on L.

defn: Let f be a linear functional on (L, τ). If f is bounded on every bounded subset of L then f is called a bounded linear functional.

Note: if (L, τ) is first countable then bounded linear functional $\Leftrightarrow$ continuous.

Note: $(L, \|\cdot\|)$ is first countable. $\Rightarrow$ boundedness is everything!

So given a linear functional $f : L \to \mathbb{R}$ we just want to know if
$\vec{0} \in U$, U bounded $\Rightarrow$ $|f(U)| \leq C < \infty$ some C (that depends on U, of course.)

Since U bounded $\Rightarrow$ $U \subseteq \{x \mid \|x\| \leq R\}$ for some R, it suffices to show that f is bounded on all balls of radius R

$$x \in \{x \mid \|x\| \leq R\} \iff x = Ry \text{ for some } y \in \{x \mid \|x\| \leq 1\}$$

$$\Rightarrow |f(x)| = |f(Ry)| = |Rf(y)| = |R|\,|f(y)|$$

so it suffices to bound f on $\{y \mid \|y\| \leq 1\}$

$\Rightarrow$ f is a bounded linear functional iff
$$\sup_{\|x\| \leq 1} |f(x)| =: \|f\| < \infty.$$

<u>defn</u>: given a bounded linear functional on $(L, \|\cdot\|)$, $\|f\|$ is the <u>norm of f</u>

<u>thm</u>: If f is a bounded linear functional on $(L, \|\cdot\|)$ then $\|f\|$ satisfies the following:

$$\|f\| = \sup_{\|x\| \neq 0} \frac{|f(x)|}{\|x\|} \quad \text{and} \quad |f(x)| \leq \|f\|\,\|x\| \;\; \forall x \in L.$$

<u>proof</u>: See K+F.

It all follows from the linearity of f and the linearity of $\|\cdot\|$ wrt scaling.

<u>ex</u>: let $a \in (L, \langle\cdot\rangle)$ $\quad \vec{a} \neq \vec{0}$.

define $f_a : L \to \mathbb{R}$ by

$$f_a(x) = \langle x, a \rangle$$

then $|f_a(x)| = |\langle x, a\rangle| \le \|x\|\,\|a\| \quad \forall x$

$$\Rightarrow \sup_{\|x\| \neq 0} \frac{|f_a(x)|}{\|x\|} \le \|a\|.$$

Moreover, $f_a(a) = \|a\|^2 \Rightarrow \displaystyle\sup_{\|x\| \neq 0} \frac{|f_a(x)|}{\|x\|} \ge \|a\|$

$$\Rightarrow \|f_a\| = \|a\|.$$

<u>ex</u>: let $w \in C([a,b])$ = cts fns on $[a,b]$ with L^∞ norm.

define $f_w : C([a,b]) \to \mathbb{R}$ by

$f_w(g) = \displaystyle\int_a^b w(x) g(x)\, dx$ then

$$|f_w(g)| \le \int_a^b |w(x)||g(x)|\, dx \le \|g\|_\infty \int_a^b |w(x)|\, dx$$

$$\Rightarrow \quad \|f_w\| \leq \int_a^b |w(x)|\,dx.$$

Now take $g \equiv 1$ on $[a,b]$.

$$f_w(g) = \int_a^b w(x)\cdot 1\,dx$$

$$|f_w(g)| = \int_a^b w(x)\,dx = \int_a^b |w(x)|\,dx \text{ if } w \geq 0.$$

$\Rightarrow$ if $w \geq 0$ then $|f_w(g)| = \int_a^b |w(x)|\,dx$

if w isn't strictly ≥ 0 or ≤ 0 then we want to test against $g = \begin{cases} +1 & \text{where } w \geq 0 \\ -1 & \text{where } w < 0 \end{cases}$

since then $f_w(g) = \int |w(x)|\,dx$,

the problem is that g isn't continuous. So we'd need to approximate g with $g_n \in C([a,b])$ so that $\|g_n - g\|_\infty \to 0$ and so that

$$\left| f_w(g_n) - \int_a^b |w(x)|\,dx \right| \to 0.$$

this would prove $\|f_w\| = \int_a^b |w(x)|\,dx.$

ex: $\delta_{x_0} : C([a,b]) \to \mathbb{R}$ $\qquad x_0 \in [a,b]$

where $\delta_{x_0}(f) = f(x_0)$

then $|\delta_{x_0}(f)| = |f(x_0)| \leq \|f\|_\infty$

$$\Rightarrow \|\delta_{x_0}(f)\| \leq 1$$

take $f \in C([a,b])$, $f \equiv 1$

then $\|\delta_{x_0}(f)\| = |f(x_0)| = 1 = \|f\|_\infty$

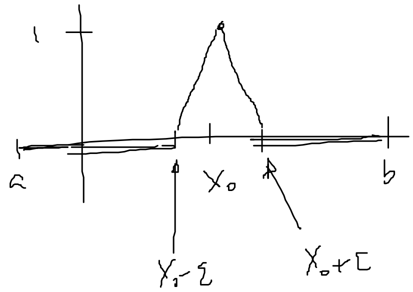
$$\Rightarrow \|\delta_{x_0}(f)\| = 1.$$

Note! If we take

$(X, \|\ \|_p) =$ completion of $C([a,b])$ with respect to the L^p norm

then δ_{x_0} is not a bounded linear functional.

Why? Consider $f_\varepsilon \in C([a,b])$ defined by



i.e. $\begin{cases} \frac{1}{\varepsilon}(x-x_0)+1 & \text{to left} \\ -\frac{1}{\varepsilon}(x-x_0)+1 & \text{to right} \\ 0 & \text{if } |x-x_0| \geq \varepsilon \end{cases}$

then $\delta_{x_0}(f_\varepsilon) = 1$ and $\|f_\varepsilon\| = \sqrt[p]{\frac{2\varepsilon}{p+1}}$

if δ_{x_0} were continuous then $\exists\, C<\infty$ such that

$$|\delta_{x_0}(f)| \leq C\|f\|_p$$

specifically, $\exists\, C$ that works for our f_ε.

i.e. $$1 = |\delta_{x_0}(f_\varepsilon)| \leq C\|f\|_p = C\sqrt[p]{\frac{2\varepsilon}{p+1}}$$

impossible if $C<\infty$ since we can take $\varepsilon \ll 1$ to violate the inequality.

<u>Note</u>: we can have linear functionals

$$f : (X, \|\cdot\|) \to (Y, |||\cdot|||)$$

in which case we'd define

$$\|f\| = \sup_{\|x\|=1} \frac{|||f(x)|||}{\|x\|}$$

I can't define X and Y rigorously (yet) but consider the functional that takes g to $g_x = \frac{\partial g}{\partial x}$

this is certainly a linear functional. (need g to have a derivative) is it bounded?

Let's consider X to contain differentiable fxns on $[0, 2\pi]$ and Y to contain continuous fxns on $[0, 2\pi]$.

specifically $g_k(x) = \cos(kx) \in X$

then $D : (X, \|\cdot\|) \to (Y, |||\ |||)$

has $D(\cos(kx)) = -k\sin(kx)$.

Bounded wrt $\|\ \| = |||\cdot||| = L^\infty$? No! Need $C < \infty$
so that

$$\|-k\sin(kx)\|_\infty \leq C \|\cos(kx)\|_\infty$$

i.e. $|k| \cdot 1 \leq C \cdot 1 \quad \forall k.$ impossible!

Bounded wrt $\|\cdot\| = |||\ ||| = L^p$? No again...

$$\sqrt[p]{\int_0^{2\pi} |-k\sin(kx)|^p dx} \leq C \sqrt[p]{\int_0^{2\pi} |\cos(kx)|^p dx}$$

i.e. $$|k| \sqrt{\int_0^{2\pi} |\sin(kx)|^p dx} \leq C \sqrt[p]{\int_0^{2\pi} |\cos(kx)|^p dx}$$

equal
since periodic

$\Rightarrow$ need $|k| \leq C \quad \forall k$ if D will be bounded.

See K+F for a geometric explanation of $\|f\|$!

let $M_f = \{x \mid f(x) = 1\}$ let $d = \text{dist}(\vec{0}, M_f)$

$d = \inf_{x \ni f(x)=1} \|x\|$. then $d = \frac{1}{\|f\|}$

Recall the Hahn-Banach theorem on vector spaces. (no $\| \|$, no τ, no $\langle , \rangle$).

There, we needed a linear functional f, a subspace L_0, and a convex functional p. on L

if $\forall x \in L_0$ we had $|f(x)| \leq p(x)$
then f could be extended to all of L and the extension would satisfy $|f(x)| \leq p(x)$ $\forall x \in L$.

thm: Hahn-Banach in $(L, \| \|)$. Let $L_0 \subseteq L$ be a subspace of L. Let f_0 be a bounded linear functional on L_0. Then f_0 can be extended to a bounded linear functional on all of L w/o increasing its norm. i.e.

$$\|f_0\|_{\text{on } L_0} = \|f\|_{\text{on } L}.$$

proof: 1) Need convex functional... use norm!

$$p(x) = \|f_0\|_{\text{on } L_0} \cdot \|x\|$$

Then $|f_0(x)| \leq \|f_0\|_{\text{on } L_0} \cdot \|x\|$ $\forall x \in L_0$.

and p is defined on all L. $\Rightarrow$ can extend f to L so that $|f(x)| \leq p(x) = \|f_0\|_{\text{on } L_0} \cdot \|x\|$ $\forall x \in L$

$\Rightarrow$ $\|f\|_{\text{on } L} \leq \|f_0\|_{\text{on } L_0}$. And $\|f\|_{\text{on } L} = \|f_0\|_{\text{on } L_0}$

because we can take a sequence in L_0 so that

$$|f(x_n)| \uparrow \|f_0\|_{\text{on } L_0} \|x_n\|$$

i.e. $\lim_{n\to\infty} |f(x_n)| = \|f_0\|_{\text{on } L_0} \|x_n\|$ for the sequence $\{x_n\} \subseteq L_0$.

Since $\{x_n\} \subseteq L$ this implies

$$\|f_0\|_{\text{on } L_0} \leq \|f\|_{\text{on } L} \Rightarrow \|f_0\|_{\text{on } L_0} = \|f\|_{\text{on } L}.$$

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See K+F for nice geometric interpretation of Hahn Banach!

Also, see K+F for the concept of (L, τ) having "sufficiently many" linear functionals. (a separation property, with a T_2 feeling to it...)