The Reisz Representation Theorem

Let X be a topological vector space and $C_c(X)$ the space of continuous real (or complex) valued functions on X with compact support.
A linear functional $\Lambda : C_c(X) \to \mathbb{R}$ (or $\mathbb{C}$) is *positive* if $f(x) \geq 0 \ \forall x \in X \Rightarrow \Lambda f \geq 0$.
i.e. $\text{range}\{f\} \subseteq [0,\infty) \Rightarrow \Lambda f \in [0,\infty)$.

Theorem (R. Repr. Theorem): Let X be a locally compact Hausdorff space and let Λ be a positive linear functional on $C_c(X)$. Then there exists a σ-algebra $\mathcal{B}$ in X which contains all Borel sets in X and there exists a unique measure μ on $\mathcal{B}$ which represents Λ in the sense that

(a) $\Lambda f = \int_X f \, d\mu$ for every $f \in C_c(X)$

and which has the following additional properties

(b) $\mu(K) < \infty$ for every compact $K \subseteq X$

(c) $\mu(\Gamma) = \inf\{\mu(V) \mid \Gamma \subseteq V, V \text{ open}\} \ \forall \Gamma \in \mathcal{B}$

(d) $\mu(E) = \sup\{\mu(K) \mid K \subseteq E, K \text{ compact}\} \ \forall E \text{ open}$

further, if $\mu(\Gamma) < \infty$ then

$$\mu(\Gamma) = \sup\{\mu(K) \mid K \subseteq \Gamma,\ K \text{ compact}\}$$

e) if $\Gamma \in \mathcal{B}$ and $A \subseteq \Gamma$ and $\mu(\Gamma) = 0$ then $A \in \mathcal{B}$.

Note: A Radon measure is a Borel measure that is finite on all compact sets, is outer regular on all Borel sets, and is inner regular on all open sets. (i.e. b), c), d) hold w/o mention of $\mu(\Gamma) < \infty \Rightarrow \mu(\Gamma) = \sup\{\ \}$. Although you can prove this is true if μ is a Radon measure.)

Note: The Reisz Representation theorem does not assume Λ is continuous. (Yikes!) Although being positive does give you something quite strong:

Proposition: If Λ is a positive linear functional on $C_c(X)$, for each compact $K \subset X$ there is a constant C_K such that

$$|\Lambda f| \leq C_K \|f\|$$

for all $f \in C_c(X)$ with $\operatorname{supp}\{f\} \subseteq K$.

The proof of uniqueness of μ is quite straightforward. From conditions c) and d) it is clear that μ is determined by its value on compact sets.

$\Rightarrow$ it suffices to prove $\mu_1(K) = \mu_2(K)$ for all K compact whenever μ_1 and μ_2 are Borel measures that satisfy the theorem.

Fix K and $\varepsilon > 0$. $\exists V$ open so that

$$\mu_2(V) < \mu_2(K) + \varepsilon < \infty$$

and $K \subseteq V$. By Urysohn's Lemma, $\exists f \in C_c(X)$ so that $\operatorname{supp}(f) \subseteq V$ and $f \equiv 1$ on K, $0 \le f \le 1$.

$$\Rightarrow \quad \mu_1(K) = \int_X 1_K \, d\mu_1 \le \int_X f \, d\mu_1 = \Lambda f = \int_X f \, d\mu_2$$

$$\le \int_X 1_V \, d\mu_2 = \mu_2(V) < \mu_2(K) + \varepsilon$$

$\Rightarrow \mu_1(K) \le \mu_2(K)$ by taking $\varepsilon \downarrow 0$.

Exchanging the roles of μ_1 and μ_2, we find

$$\mu_2(K) \le \mu_1(K) \quad \Rightarrow \mu_1(K) = \mu_2(K)$$

$\forall K$ compact. $\Rightarrow \mu$ is uniquely det'd.

See Rudin's "Real and Complex Analysis" pp 40-47 for a proof.

Or see Folland's "Real Analysis" pp 212 - 215

Or see Stroock's book chapter 8.

If you're feeling abstract, you can use the Reisz-Representation theorem to define Lebesgue measure. (Look Mom! No construction!)

Step 1: Define your Λ on $C_c(\mathbb{R}^n)$.

$$\Lambda f := \int_{\mathbb{R}^n} f(x)\,dx \longleftarrow \text{the } \underline{\underline{\text{Riemann}}} \text{ Integral}$$

Step 2: prove translation invariance of μ and the fact that μ respects linear transformations. All of this follows from the Riemann Integral's doing this and being patient.