We want to prove some density results.

<u>Theorem</u> for $1 \le p < \infty$, the set of simple functions

$$\phi = \sum_1^n a_j 1_{E_j} \quad \text{when } \mu(E_j) < \infty \ \ j = 1 \dots n, \ |a_j| < \infty$$

is dense in $L^p(\mu)$.

<u>Proof:</u> Fix $f \in L^p$. As before, we define

$$\phi_n = \sum_{k=0}^{4^n - 1} \frac{k}{2^n} 1_{\{f \in [k/2^n, (k+1)/2^n)\}} + 2^n 1_{\{f \ge 2^n\}}$$

$$+ \sum_{k=1-4^n}^{0} \frac{k}{2^n} 1_{\{f \in (\frac{k-1}{2^n}, \frac{k}{2^n}]\}} - 2^n 1_{\{f \le -2^n\}}$$

ϕ_n is certainly simple. It's not clear that $\mu(E_j) < \infty$ for $j = 1 \dots 4^{n+1}$ but we'll return to that.

We know that $\phi_n \to f$ pointwise and

$|\phi_n| \le |f|$ for all n and $|\phi_n| \le |\phi_{n+1}| \le \dots$

Since $|\phi_n| \le |f|$ and $\int_E |f|^p d\mu < \infty$ we know

$\int_E |\phi_n|^p d\mu < \infty$ for each n. Since $\phi_n = \sum_j^{4^{n+1}} a_j 1_{E_j}$ with $E_j \cap E_\ell = \emptyset$ for $j \ne \ell$

We see $|\phi_n|^p = \sum_{j=1}^{4^{n+1}} |a_j|^p \mathbb{1}_{E_j}$

$$\Rightarrow \int_{E_j} |\phi_n|^p = |a_j|^p \mu(E_j) \leq \int_E |\phi_n|^p < \infty \Rightarrow \mu(E_j) < \infty \text{ for each } j.$$

$\Rightarrow$ Our simple functions ϕ_n are in the right class.
We just need to show $\|\phi_n - f\|_{L^p} \to 0$ as $n \to \infty$.

This follows from Lebesgue dominated convergence

since $$|\phi_n - f|^p \leq (|\phi_n| + |f|)^p \leq (|f| + |f|)^p = 2^p |f|^p \in L^1$$

so $|\phi_n - f|^p \to 0$ pointwise and $|\phi_n - f| \leq g \in L^1$

$$\Rightarrow \lim_{n\to\infty} \int |\phi_n - f|^p d\mu = \int \lim_{n\to\infty} |\phi_n - f|^p d\mu = 0$$

$$\Rightarrow \|\phi_n - f\| \to 0 \text{ as } n \to \infty.$$

This finishes the proof. //

Theorem: The simple functions are dense in $L^\infty(\mu)$.

WHOAH! Before we had "simple functions w/ finite measure support" Now we don't. Is that real? Yes. Consider $L^\infty(\mathbb{R})$ and $f = 1$ then you can't approx. w/ simple fns w/ finite-measure support.

Proof: Let f be an L^∞ function.

Since f is measurable, we know that our ϕ_n functions converge pointwise to f

If f were bounded then they would converge uniformly to f in the $\mathbb{R}$-topology.

Since f is in L^∞, we know f is essentially bounded, i.e.

$$\mu(x \mid |f(x)| > \|f\|_\infty) = 0.$$

Let $E_1 = \{f \le \|f\|_\infty\}$. f is bounded on $E_1 \Rightarrow \phi_n$ converges uniformly to f on E_1. Fix $\varepsilon > 0$. $\Rightarrow \exists N_\varepsilon$ so that $n \geq N_\varepsilon \Rightarrow \sup_{x \in E_1} |\phi_n(x) - f(x)| < \varepsilon \Rightarrow \mu\{|\phi_n - f| \geq \varepsilon\} = 0$

$\Rightarrow$ ess-sup $|\phi_n - f| \le \varepsilon \Rightarrow \|\phi_n - f\|_\infty \le \varepsilon$, as desired.

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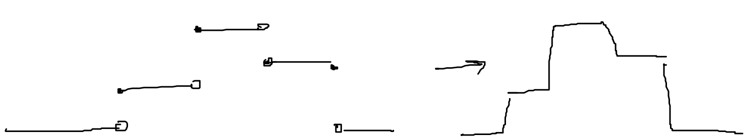
Now if we were looking at $L^p(\mathbb{R})$ then it would be obvious that continuous functions with compact support are dense if $1 \le p < \infty$. Why? Find a step function that is ε-close and make it continuous:



← Continuous + compact support.

Q1: what if it's $L^p(\mu)$ rather than $L^p(\mathbb{R})$?

Q2: what if it's L^∞?

Consider $(E, \mathcal{B}, \mu)$ such that

1) E is a locally compact Hausdorff space
2) the measure μ satisfies:
   a) $\mu(K) < \infty$ for every compact set $K \subset E$
   b) if $\Gamma \in \mathcal{B}$ then $\mu(\Gamma) = \inf\{\mu(V) \mid \Gamma \subseteq V, V \text{ open}\}$
   c) if Γ is open then $\mu(\Gamma) = \sup\{\mu(K) \mid K \subseteq \Gamma, K \text{ compact}\}$
      also, if $\mu(\Gamma) < \infty$ then $\mu(\Gamma) = \sup\{\mu(K) \mid K \subseteq \Gamma, K \text{ compact}\}$.
   d) if $\Gamma \in \mathcal{B}$ and $\Gamma_0 \subseteq \Gamma$ and $\mu(\Gamma) = 0$ then $\Gamma_0 \in \mathcal{B}$.

d) is that $(E, \mathcal{B}, \mu)$ is a complete measure space.

c) is μ is "inner regular"

b) is μ is "outer regular"

Note: $\mathbb{R}^n$ with the Borel sets and Lebesgue measure satisfies our requirements

Recall Urysohn's lemma:

Suppose E is a locally compact Hausdorff space, V is an open set, $K \subseteq V$ is compact. Then there is a continuous real-valued function with compact

support, $f \in C_c(E)$, such that

$$f(x) = 1 \quad \forall x \in K$$
$$\operatorname{supp}(f) \subseteq V$$
$$0 \leq f(x) \leq 1 \quad \forall x \in E$$

We proved this in the Fall semester.

Lusin's theorem: Suppose f is a complex (or real) measurable function on E, $A \subseteq E$ and $\mu(A) < \infty$ and $f(x) = 0$ if $x \notin A$. Assume $(E, \mathcal{B}, \mu)$ satisfy conditions 1) & 2). Then there exists $g \in C_c(E)$ such that

$$\mu(\{x \mid f(x) \neq g(x)\}) < \varepsilon.$$

Furthermore, we may arrange it so that

$$\sup_{x \in E} |g(x)| \leq \sup_{x \in E} |f(x)|.$$

Proof: First, let's assume $0 \le f \le 1$ and A is compact.

Let ϕ_n be our (usual) sequence of simple functions

$$\phi_n = \sum_{k=0}^{2^{n}-1} \frac{k}{2^n} 1_{f \in [k/2^n,\, (k+1)/2^n)} + 1_{f=1}$$

$$\phi_{n+1} = \sum_{k=0}^{2^{n+1}-1} \frac{k}{2^{n+1}} 1_{f[k/2^{n+1},\, (k+1)/2^{n+1})} + 1_{f=1}$$

note: $\left\{f \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right\} = \left\{f \in \left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right)\right\} \cup \left\{f \in \left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right)\right\}$

$$\Rightarrow \phi_{n+1} - \phi_n = \sum_{k=0}^{2^n-1} \left(\frac{2k+1}{2^{n+1}} - \frac{k}{2^n}\right) 1_{f \in \left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right)}$$

$$\phi_{n+1} - \phi_n = \frac{1}{2^{n+1}} \sum_{k=0}^{2^n-1} 1_{f \in \left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right)}$$

let $\psi_1 = \phi_1$

$\psi_n = \phi_n - \phi_{n-1}$ for $n \geq 2$

$$\Rightarrow 2^n \psi_n = 1_{E_n} \quad \text{for some set } E_n \subseteq A.$$

and since $\lim_{n\to\infty} \phi_n(x) = f(x)$ for each x,

we know $\sum_{n=1}^{\infty} \psi_n(x) = f(x)$ for each x.

Fix an open set V such that $A \subset V$ and $[V]$ is compact. (Why does such a V exist? see lemma at end of proof.) Then there are compact sets K_n and open sets V_n such that $K_n \subseteq E_n \subseteq V_n \subseteq V$ and $\mu(V_n - K_n) \leq \frac{\varepsilon}{2^n}$

Why? see lemma at end of proof.

By Urysohn's lemma there exists $h_n \in C_c(E)$ such that

$$h_n(x) = 1 \quad \forall x \in K_n$$
$$\operatorname{supp}(h_n) \subseteq V_n$$
$$0 \leq h_n \leq 1$$

Define $g(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x) \qquad x \in E$,

The series converges uniformly on E since given $\varepsilon > 0$, take N so that $n \geq N \Rightarrow \sum_{n}^{\infty} 2^{-k} < \varepsilon$.

$$\Rightarrow \sum_{k}^{\infty} 2^{-k} h_k(x) \leq \sum_{n}^{\infty} 2^{-n} < \varepsilon \quad \text{since} \quad 0 \leq h_k \leq 1 \ \forall k.$$

Since the convergence is uniform, we know g is continuous on E. Also,

$$\operatorname{supp}(g) \subseteq \bigcup_{1}^{\infty} \operatorname{supp}(h_k) \subseteq \left[\bigcup_{1}^{\infty} V_n\right] \subseteq [V]$$

$\therefore$ The support of g lies in a compact set.

$\Rightarrow g \in C_c(E)$. Now to show that

$$\mu\{x \mid f(x) \neq g(x)\} < \varepsilon.$$

We know $\quad 2^{-n} h_n = \psi_n \quad$ in K_n

and we know $\operatorname{supp}(h_n) \subset V_n$ and
$\operatorname{supp}(\psi_n) \subseteq V_n$.

$\Rightarrow \quad 2^{-n} h_n = \psi_n$ except in $V_n - K_n$

since $f(x) = \sum_{1}^{\infty} \psi_n(x) \quad \forall x \in E$

we know $f(x) = g(x)$ except in at worst

$$\bigcup_{n=1}^{\infty} (V_n - K_n) \quad \Rightarrow \mu(x \mid f(x) \neq g(x)) \leq \mu\left(\bigcup_{1}^{\infty} (V_n - K_n)\right) < \varepsilon$$

as desired.

If A is compact and f is <u>bounded</u> $m \leq f(x) \leq M$ $\forall x \in E$ then we add m to f and then divide by $(M-m)$

If A isn't compact, but $\mu(A) < \infty$ then we start by choosing $K \subseteq A$ with K compact and $\mu(A \sim K) < \varepsilon$. We then do the construction on the set K and find $\mu(f \neq g) < 2\varepsilon$. (Why can we find this K? Recall the inner regularity

$$\mu(A) < \infty \Rightarrow \mu(A) = \sup\{\mu(K) \mid K \subseteq A,\ K \text{ compact}\}.$$

If f is a real measurable function and if $B_n = \{x \mid |f(x)| > n\}$ then $\bigcap_1^\infty B_n = \emptyset$ (since we didn't allow f to take values at $\pm\infty$) and so $\mu(B_n) \to 0$ as $n \to \infty$. So given $\varepsilon > 0$, choose n so that $\mu(B_n) < \varepsilon$. Now take $\tilde{f} = 1_{B_n^c} \cdot f$ $\tilde{f}$ is a <u>bounded</u> measurable function by construction. $\mu(f \neq \tilde{f}) < \varepsilon$. Now

$\operatorname{supp}(\tilde{f}) \subseteq \underbrace{B_n \cup A}$ this set has finite measure $\Rightarrow$ we can find $g \in C_c(E)$ so that $\mu(g \neq \tilde{f}) < \varepsilon$

$\Rightarrow \mu(g \neq f) < 2\varepsilon$

Finally, let $R = \sup\{ |f(x)| \mid x \in E \}$
and define $\phi(z) = z$ if $|z| \le R$
$= Rz/|z|$ if $|z| > R$.

Then ϕ is a continuous mapping of $\mathbb{R}$ to the interval $[-R, R]$. Furthermore if

$$\mu(f \ne g) < \varepsilon$$

then taking $\tilde{g} = \phi \circ g$ we have

$$\mu(f \ne \tilde{g}) < \varepsilon \quad \text{and} \quad \sup|\tilde{g}| \le \sup|f|. \checkmark$$

$\ulcorner$ if $x \in \{f = g\}$ then $|g(x)| \le R \Rightarrow \tilde{g}(x) = g(x)$
$\Rightarrow \tilde{g}(x) = f(x)$
$\Rightarrow x \in \{f = \tilde{g}\}$.

$$\Rightarrow \{f \ne \tilde{g}\} \subseteq \{f \ne g\} \Rightarrow \mu(f \ne \tilde{g}) \le \mu(f \ne g) < \varepsilon. \lrcorner$$

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This finishes the proof of Lusin's theorem. I owe you two lemmas

<u>lemma 1</u>: Let A be compact. Then $\exists$ an open set V such that $A \subseteq V$ and $[V]$ is compact.

<u>proof</u>:
For each $x \in A$ $\exists$ an open set U_x s.t. that $x \in U_x$ and $[U_x]$ is compact.
$A \subset \bigcup_x (U_x)$. Since A is compact, $\exists x_1 \dots x_k$ so that $A \subseteq \bigcup_1^k U_{x_i}$ and $V := \bigcup_1^k U_{x_i}$ is open. Furthermore, since it's a <u>finite</u> union, $[V]$ is compact. //

<u>lemma 2</u>: Let E be measurable set w/ $\mu(E) < \infty$. Then $\exists$ compact set K and open set V such that $K \subseteq E \subseteq V$ and $\mu(V - E) < \varepsilon$

<u>proof</u>: By the outer regularity,

$$\mu(E) = \inf\{\mu(V) \mid E \subseteq V,\ V \text{ open}\}.$$

$\Rightarrow \exists\, V$ so that $\mu(V) \leq \mu(E) + \varepsilon/2$

since $\mu(E)<\infty$, the inner regularity tells us

$$\mu(E) = \sup\{\mu(K) \mid K\subseteq E, K \text{ compact}\}$$

$\Rightarrow \exists K$ so that $K \subseteq E$

$$\mu(K) > \mu(E) - \varepsilon/2.$$

$$\mu(V-E) = \mu(V) - \mu(E) \leq \mu(E) + \varepsilon/2 - \mu(E) = \varepsilon/2$$

$$\mu(E-K) = \mu(E) - \mu(K) \leq \mu(E) - \mu(E) + \varepsilon/2 = \varepsilon/2$$

$\rightarrow \mu(V-K) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$ //

Now we're ready to prove our density theorem!

<u>theorem</u>: Assume E is a locally compact Hausdorff space and μ has the properties 1) & 2). For $1\leq p<\infty$, $C_c(X)$ is dense in $L^p(\mu)$.

<u>proof</u>: Fix $\varepsilon>0$. Let ψ be a simple function
$$\psi = \sum_{j=1}^{n} \alpha_i 1_{E_i} \quad \text{where } \mu(E_i)<\infty \quad i=1..n.$$
Since $\exists A$ so that $\mu(A)<\infty$ and $\psi(x)=0$ if $x\notin A$, Lusin's Theorem applies.

$\Rightarrow \exists\ g \in C_c(E)$ so that $g(x) = \psi(x)$ for all x outside a set of measure $< \varepsilon$. and $|g| \leq \|\psi\|_\infty$. $\Rightarrow$

$$\int_E |g(x) - \psi(x)|^p \, d\mu(x) = \int_{\{g \neq \psi\}} |g - \psi|^p \, d\mu(x)$$

$$\leq \int_{\{g \neq \psi\}} 2^p |\psi|^p \, d\mu(x)$$

$$\leq 2^p \|\psi\|_\infty^p \int_{\{g \neq \psi\}} d\mu(x) = 2^p \varepsilon \|\psi\|_\infty^p.$$

$$\Rightarrow \|g - \psi\|_p \leq 2 \sqrt[p]{\varepsilon} \, \|\psi\|_\infty.$$

This shows $C_c(E)$ is dense in the set of simple functions w/ $\sum_1^n \mu(E_i) < \infty$. Since these functions are dense in $L^p(\mu)$ for $1 \leq p < \infty$, we're done. //

Note: We've just proven that $L^p(\mu)$ is the completion of the metric space from the beginning of the course:

$$X = \{ f \mid f \text{ continuous on } [a,b] \},$$

wrt $\|\cdot\|_p$ defined via Riemann integrals.

Also, we've shown that

$L^p(\mathbb{R}^n)$ = completion of the continuous functions w/ compact support.

So we don't have to worry about improper integrals. We take Riemann integrals to define the metric $\|\cdot\|_p$ on the space of compactly supported continuous functions and then take the completion.

What about $L^\infty(\mathbb{R}^n)$?

Note: the above density theorems prove that $L^p(\mathbb{R}^n)$ is separable for $1 \le p < \infty$. We know $L^\infty(\mathbb{R})$ isn't separable (from HW)

So we cannot have that

$$\left[C_c(\mathbb{R}^n)\right]\Big|_{\text{wrt } \|\cdot\|_\infty} \overset{\text{imposs!}}{=} L^\infty(\mathbb{R}^n)$$

We need control on the functions at infinity.

<u>Defn</u>: A real-valued function on a locally compact Hausdorff space E is said to <u>vanish at infinity</u> if to each $\varepsilon > 0$ there is a compact set K_ε such that $|f(x)| < \varepsilon$ for all $x \notin K_\varepsilon$.

The class of all continuous functions on E that vanish at infinity is denoted $C_0(E)$.

Clearly $C_c(E) \subseteq C_0$.

Theorem: Let E be a locally compact Hausdorff space. Then $C_0(E)$ is the completion of $C_c(E)$ with respect to the sup-norm

$$\|f\| = \sup_{x \in E} |f(x)|$$

<u>Note</u>: No measures anywhere. The sup norm is stronger than $\|\cdot\|_\infty$ and they coincide for functions in $C_c(E)$

<u>proof</u>: $C_0(E)$ is a metric space wrt $\|\ \|$.

if $\rho(f,g) := \|f-g\|$. We want to show

1) $C_c(E)$ is dense in $C_0(E)$ 2) $C_0(E)$ is a

complete metric space. (Note: the second is important to do! For example, if we take our space to be $(0,1)$ and we show that $\mathbb{Q} \cap (0,1)$ is dense in $(0,1)$ this doesn't show that $[\mathbb{Q} \cap (0,1)] = (0,1)$! In fact it just shows $(0,1) \subseteq [\mathbb{Q} \cap (0,1)]$ ← the completion of our dense subset.)

Given $f \in C_0(E)$ and $\varepsilon > 0$ $\exists$ K compact so that $|f(x)| < \varepsilon$ $\forall x \notin K$. Urysohn's lemma gives us a function $g \in C_c(E)$ such that $0 \leq g \leq 1$ and $g(x) = 1$ on K. Define $h(x) = g(x) f(x)$. Then $h \in C_c(E)$ and

$$|f(x) - h(x)| = 0 \quad \text{if } x \in K$$

$$\begin{aligned} |f(x) - h(x)| &= |f(x) - g(x) f(x)| \\ &\leq |f(x)| \, |1 - g(x)| \\ &\leq |f(x)| \leq \|f\| \quad \text{if } x \notin K \end{aligned}$$

Either way, $\|f - h\| < \varepsilon$. This proves that $C_c(E)$ is dense in $C_0(E)$.

Let $\{f_n\}$ be a Cauchy sequence in $C_0(E)$.

$\Rightarrow \{f_n\}$ converges uniformly to a limit function f. This pointwise limit is continuous.

Given $\varepsilon > 0$, $\exists$ n so that $\|f_n - f\| < \varepsilon/2$.

And $\exists$ a compact set K so that $|f_n(x)| < \varepsilon/2$ for $x \notin K$. $\Rightarrow$ $|f(x)| \leq |f_n(x)| + |f - f_n(x)| < \varepsilon$ for $x \notin K$ $\Rightarrow f \in C_0(E)$. This proves $(C_0(E), \|\cdot\|)$ is a complete metric space

w/ $C_c(E)$ dense in it. //