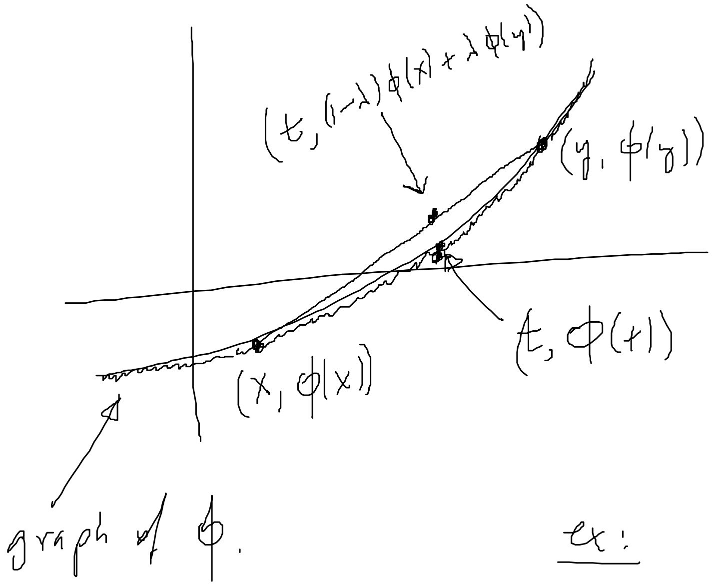
We'll use functional analysis to prove L^1 is complete, rather than following in Stroock's path.

<u>defn</u>: A real function ϕ defined on a segment (a,b) where $-\infty \le a < b \le \infty$ is called <u>convex</u> if the inequality $\phi((1-\lambda)x + \lambda y) \le (1-\lambda)\phi(x) + \lambda\phi(y)$ holds whenever $x, y \in (a,b)$ and $\lambda \in [0,1]$.



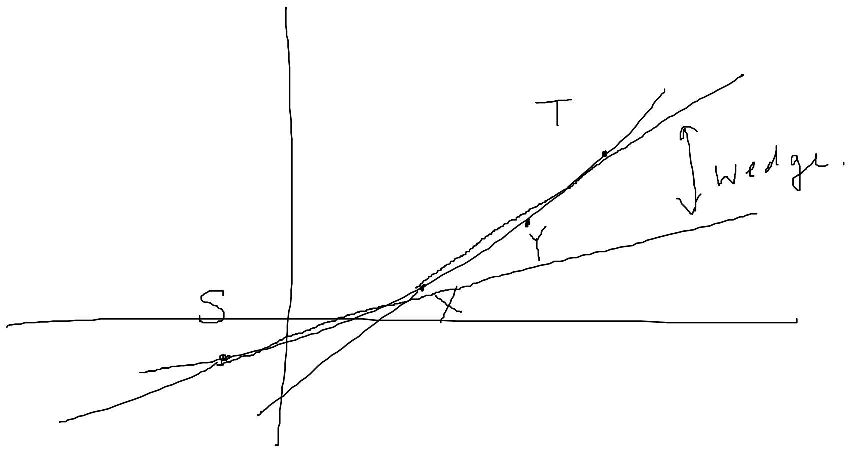
<u>ex</u>: $x \to e^x$ is convex on $\mathbb{R}$.

<u>theorem</u>: if ϕ is convex on (a,b) then ϕ is continuous on (a,b)

<u>proof</u>: Suppose $a < s < x < y < t < b$. Let $S \in \mathbb{R}^2$ be the point $(s, \phi(s))$, $X = (x, \phi(x))$, $Y = (y, \phi(y))$, $T = (t, \phi(t))$.

Then X is on or below the line connecting S & Y

hence Y is on or above the line through S and X.
also, Y is on or below the line through X and T.



Y is trapped in the wedge $\Rightarrow$ as $y \downarrow x$, $Y \to X$
hence $\phi(y) \to \phi(x)$. Left hand limits are dealt with similarly. $\Rightarrow \lim_{y \to x} \phi(y) = \phi(x) \Rightarrow \phi$ is cts at x //

<u>thm:</u> (Jensen's Inequality) Let μ be a measure on a σ-algebra $\mathcal{B}$ on a set E and assume $\mu(E) = 1$. if f is a real-valued function in $L^1(\mu)$ and if $a < f(x) < b$ for all $x \in E$ and if ϕ is convex on (a, b) then

$$\phi\left(\int_E f\,d\mu\right) \le \int_E \phi(f)\,d\mu$$

Note: It could be that $f \in L^1(\mu)$ but $\phi(f) \notin L^1(\mu)$.
This isn't a problem since then the RHS $= \infty$.

proof: Let $t = \int_E f \, d\mu$. Then $a < t < b$ (since
$a < f < b \Rightarrow \int_E a < \int_E f \, d\mu < \int_E b \, d\mu = b$ since $\mu(E) = 1$.)

Since ϕ is convex,

$$\frac{\phi(t) - \phi(s)}{t-s} \leq \frac{\phi(u) - \phi(t)}{u-t}$$

for any $a < s < t < u < b$.

Let $\beta = \sup_{s \in (a,t)} \frac{\phi(t) - \phi(s)}{t-s}$. Then $\beta \leq \frac{\phi(u) - \phi(t)}{u-t}$ $\forall u \in (t,b)$

$\Downarrow$

$\beta(t-s) \geq \phi(t) - \phi(s)$

$\Rightarrow \phi(s) \geq \phi(t) + \beta(s-t)$

$\forall s \in (a,t)$

$\Downarrow$

$\beta(u-t) \leq \phi(u) - \phi(t)$

$\phi(t) + \beta(u-t) \leq \phi(u)$

$\forall u \in (t,b)$

$$\Rightarrow \quad \phi(t) + \beta(u-t) \leq \phi(u) \quad \forall u \in (a,b).$$

Since $f(x) \in (a,b)$ for each $x \in E$,
$\phi(t) + \beta(f(x) - t) \leq \phi(f(x))$ for each $x \in E$.

Since ϕ is continuous, $\phi \circ f$ is measurable.

$\Rightarrow \int_E \phi \circ f \, d\mu$ is defined.

$$\Rightarrow \int_E \phi(t) + \beta(f(x) - t) \, d\mu \leq \int_E \phi(f(x)) \, d\mu$$

LHS: $\int_E \phi(t) \, d\mu + \beta \int_E f(x) - t \, d\mu$

$= \phi(t) \cdot 1 + \beta \int_E f(x) - \beta t$ since $\mu(E)$.

recall $t := \int f(x) \, d\mu \Rightarrow$ LHS $= \phi(t)$

$= \phi\left(\int f(x) \, d\mu \right)$

$$\Rightarrow \phi\left(\int f \, d\mu \right) \leq \int \phi(f) \, d\mu \quad //$$

Corr: $\int_0^1 |f(x)| \, dx \leq \sqrt{\int_0^1 |f(x)|^2 \, dx} \quad \forall f \in L^1$

why? take $\phi(x) = x^2$

Corr: $\sqrt[p]{\int_0^1 |f(x)|^p \, dx} \leq \sqrt[q]{\int |f(x)|^q \, dx} \quad \forall f \in L^p$ if $q > p$

why? take $\phi(x) = x^{q/p}$

Note: if $\int_3^{10} \dots dx$ then just change variables so that the domain has unit measure and then apply Jensen.

Theorem: Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $(E, \mathcal{B}, \mu)$ be a measure space and f and g are measurable $[0,\infty]$-valued functions on E. Then

$$\int_E fg \, d\mu \leq \sqrt[p]{\int_E f^p d\mu} \sqrt[q]{\int_E g^q d\mu} \quad (*)$$

and

$$\sqrt[p]{\int_E (f+g)^p d\mu} \leq \sqrt[p]{\int_E f^p d\mu} + \sqrt[q]{\int_E g^q d\mu}$$

Recall Hölder, Schwartz, and Minkowski from the first week of class.

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proof: Let $A := \sqrt[p]{\int_E f^p d\mu}$ and $B := \sqrt[q]{\int_E g^q d\mu}$

if $A = 0$ then $f = 0$ almost everywhere and both inequalities hold. If $A > 0$ and $B = \infty$ then $(*)$ is again trivial.

So we can safely assume $A > 0$ and $B < \infty$

Similarly, we can assume $B > 0$ and $A < \infty$

Let $F := f/A$ and $G := g/B$. then $\int F^p d\mu = 1$ and $\int G^q d\mu = 1$.

If $x \in E$ is such that $0 < F(x) < \infty$ and $0 < G(x) < \infty$ then $\exists$ $s \& t \in \mathbb{R}$ $\ni$ $F(x) = e^{s/p}$ and $G(x) = e^{t/q}$.

Since $\frac{1}{p} + \frac{1}{q} = 1$ and since e^x is convex on $(-\infty, \infty)$,

$$e^{\frac{1}{p}s + \frac{1}{q}t} \leq \frac{1}{p} e^s + \frac{1}{q} e^t$$

$$\Rightarrow \quad F(x)\,G(x) \leq \frac{1}{p}\,(F(x))^p + \frac{1}{q}\,(G(x))^q$$

Let $E_1 = \{x \mid F(x) = 0 \text{ or } G(x) = 0\}$

$E_2 = \{x \mid F(x) = \infty\}$

$E_3 = \{x \mid G(x) = \infty\}$

$E_4 = \{x \mid 0 < F(x), G(x) < \infty\}$

Since we assumed $A, B < \infty$, we know $\mu(E_2) = \mu(E_3) = 0$.

$$\int_E FG\,d\mu = \int_{E_1 \cup E_4} FG\,d\mu = \int_{E_4} FG\,d\mu$$

$$\le \int_{E_4} \frac{1}{p} F^p + \frac{1}{q} G^p\,d\mu$$

$$\le \int_E \frac{1}{p} F^p + \frac{1}{q} G^p\,d\mu = \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \int \frac{f}{\|f\|_p} \frac{g}{\|g\|_q}\,d\mu \le 1 \Rightarrow \int fg\,d\mu \le \|f\|_p \|g\|_q$$ as desired. ✓

(Note: before, we used Young's inequality. This proof has the advantage that you don't need to prove Young's inequality.)

Now to prove

$$\sqrt[p]{\int_E (f+g)^p\,d\mu} \le \sqrt[p]{\int_E f^p\,d\mu} + \sqrt[p]{\int_E g^p\,d\mu}$$

$(f+g)^p = f(f+g)^{p-1} + g(f+g)^{p-1}$ 7

By Hölder's inequality,

$$\int_E f(f+g)^{p-1}\,d\mu \le \sqrt[p]{\int_E f^p\,d\mu}\;\sqrt[q]{\int_E (f+g)^{q(p-1)}\,d\mu}$$

$$\int_E g(f+g)^{p-1}\,d\mu \le \sqrt[p]{\int_E g^p\,d\mu}\;\sqrt[q]{\int_E (f+g)^{q(p-1)}\,d\mu}$$

adding,

$$\int (f+g)^p\,d\mu \le \sqrt[q]{\int_E (f+g)^p\,d\mu}\left\{\sqrt[p]{\int_E f^p\,d\mu} + \sqrt[p]{\int g^p\,d\mu}\right\}$$

If $\sqrt[p]{\int (f+g)^p\,d\mu} = 0$ then Minkowski is trivially true.

If $\sqrt[p]{\int f^p\,d\mu} + \sqrt[p]{\int g^p\,d\mu} = \infty$ then Minkowski is trivially true.

So assume $\sqrt[p]{\int (f+g)^p\,d\mu} > 0$ and $\sqrt[p]{} + \sqrt[p]{} < \infty$.

Since $x \to x^p$ is a convex function $\left(\frac{f+g}{2}\right)^p < \frac{1}{2}(f^p + g^p)$

when $f, g < \infty$. $\Rightarrow$ $\sqrt[p]{\int f^p\,d\mu} + \sqrt[p]{\int g^p\,d\mu} < \infty \Rightarrow \int (f+g)^p\,d\mu < \infty$

Since

$(f+g)^p = 2^p\left(\frac{1}{2}f + \frac{1}{2}g\right)^p \le 2^p\left[\frac{1}{2}f^p + \frac{1}{2}g^p\right] = 2^{p-1}\left[f^p + g^p\right]$

$\Rightarrow \int_E (f+g)^p \le 2^{p-1}\int_E f^p\,d\mu + 2^{p-1}\int_E g^p\,d\mu$. RHS $< \infty \Rightarrow$ LHS $< \infty$

Since $\int_E (f+g)^p \, d\mu < \infty$ we can safely divide

both sides of

$$\int_E (f+g)^p \, d\mu \leq \sqrt[q]{\int_E (f+g)^p \, d\mu} \left\{ \sqrt[p]{\int f^p \, d\mu} + \sqrt[p]{\int g^p \, d\mu} \right\}$$

by $\sqrt[q]{\int_E (f+g)^p \, d\mu}$ and we're done! //

the L^p spaces await us...

<u>defn:</u> If $0 < p < \infty$ and f is a complex-valued measurable function on $(E, \mathcal{B}, \mu)$ we define

$$\|f\|_p := \sqrt[p]{\int_E |f|^p \, d\mu}$$

and we let

$$L^p(\mu) = \text{all } f \text{ for which } \|f\|_p < \infty.$$

We call $\|f\|_p$ the L^p-norm of f.

<u>defn</u>: Suppose $g: E \to [0,\infty]$ is measurable.

Let $S = \{ \alpha \in \overline{\mathbb{R}} \mid \mu(g^{-1}((\alpha,\infty])) = 0 \}$.

If $S = \phi$ then $\beta := \infty$. If $S \neq \phi$ then $\beta := \inf S$

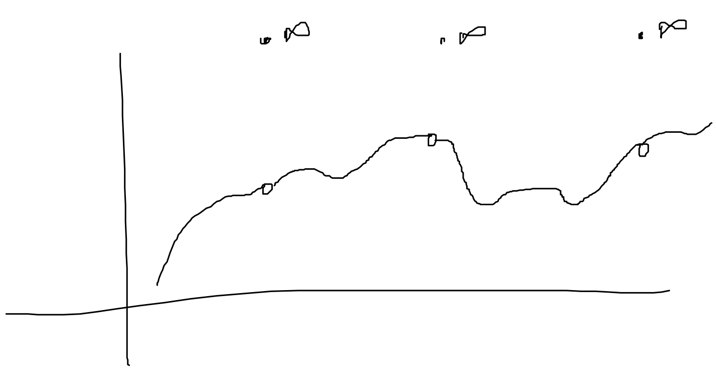
Since

$$g^{-1}((\beta,\infty]) = \bigcup_{n=1}^{\infty} g^{-1}((\beta+\tfrac{1}{n},\infty])$$

and since the countable union of sets of measure zero is also a set of measure zero, we see $\beta \in S$.

β is the <u>essential</u> <u>supremum</u> of g.

<u>defn</u>: If f is a complex-valued measurable function on $(E, \mathcal{B}, \mu)$, we define $\|f\|_\infty$ to be the essential supremum of $|f|$ and let $L^\infty(\mu)$ be all those functions for which $\|f\|_\infty < \infty$.



← this is the graph of an L^∞ function

<u>theorem</u>: if $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ and

if $f \in L^p(\mu)$ and $g \in L^q(\mu)$ then

$fg \in L^1(\mu)$ and

$$\int_E |fg| \, d\mu \le \sqrt[p]{\int_E |f|^p \, d\mu} \; \sqrt[q]{\int_E |g|^q \, d\mu}$$

<u>proof</u>: For $1 < p < \infty$, this is just Hölder's inequality applied to $|f|$ and $|g|$.

If $p = \infty$ then $|f(x) g(x)| \le \|f\|_\infty |g(x)|$ for almost all x.

Integrating, we get

$$\int_E |fg| \, d\mu \le \|f\|_\infty \|g\|_1 \quad \text{and done.}$$

If $p = 1$ then repeat above argument, reversing the roles of f and g.

//

<u>theorem</u>: Suppose $1 \le p \le \infty$, $f \in L^p(\mu)$ and $g \in L^p(\mu)$.

then $f + g \in L^p(\mu)$ and $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

Proof: if $1<p<\infty$ then this is just Minkowski.
If $p=1$ or ∞ then it follows immediately from $|f(x)+g(x)| \le |f(x)|+|g(x)|$. //

If $\alpha \in \mathbb{C}$ then $\|\alpha f\|_p = |\alpha|\,\|f\|_p$.

The above theorems show that $L^p(\mu)$ is a complex vector space.

Further, if we assume $1 \le p \le \infty$ and we introduce equivalence classes, then $L^p(\mu)$ is a normed vector space.

Now we want to show $L^p(\mu)$ is a Banach space.
I'll present two proofs.

Theorem: For $1 \le p < \infty$, $L^p(\mu)$ is a Banach space.

Proof: Suppose $\{f_k\}_1^\infty \subseteq L^p(\mu)$ and $\underbrace{\sum_1^\infty \|f_k\|_p}_{=B} < \infty$.
We'll prove that $\sum_1^\infty f_k = f \in L^p(\mu)$. This will (by earlier theorem) prove $L^p(\mu)$ is complete.

Let $G_n = \sum_1^n |f_k|$ and $G = \sum_1^\infty |f_k|$

By Minkowski's inequality, $G_n \in L^p(\mu)$ and

$\|G_n\|_p \le \sum_1^n \|f_k\|_p \le B$. is true $\forall n$.

By the monotone convergence theorem, since

$G_1 \le G_2 \le G_3 \le \dots$ and $\lim_{n\to\infty} G_n = G$, we know

$G_1^p \le G_2^p \le G_3^p \le \dots$ and $\lim_{n\to\infty} G_n^p = G^p$ and

hence $\lim_{n\to\infty} \int G_n^p \, d\mu = \int \lim_{n\to\infty} G_n^p \, d\mu \le B^p$

This proves that $G = \sum_1^\infty |f_k| \in L^p(\mu) \Rightarrow G(x) < \infty$ almost everywhere in μ. $\Rightarrow \sum_1^\infty f_k$ converges almost everywhere in μ. We want the limit in L^p B conv. in L^p.

Let $F = \sum_1^\infty f_k$. We know $|F| \le G \Rightarrow F \in L^p(\mu)$.

$|F - \sum_1^n f_k|^p \le (|F| + |\sum_1^n f_k|)^p \le (2G)^p \in L^1(\mu)$.

$\Rightarrow$ We can use $(2G)^p$ as an upper bound in Lebesgue Dominated convergence theorem and therefore

$$\lim_{n\to\infty} \int |F - \sum_1^n f_k|^p \, d\mu = \int \lim_{n\to\infty} |F - \sum_1^n f_k|^p$$

the RHS $\to 0$ as $n \to \infty$ $\Rightarrow \|F - \sum_1^n f_k\|_p \to 0$ as desired. //

Here's an alternate proof.

<u>theorem</u>: $L^p(\mu)$ is a complete metric space for $1 \le p \le \infty$.

<u>corr</u>: If $1 \le p \le \infty$ and if $\{f_n\}_1^\infty$ is a Cauchy sequence in $L^p(\mu)$ with limit f, then $\{f_n\}$ has a subsequence which converges pointwise almost everywhere to f.

(proof of corr is in the proof of the theorem. Dig it out!)

<u>Proof of theorem</u>:

<u>case 1</u>: $(1 \le p < \infty)$

Since $\{f_n\}$ is Cauchy, we can find a subsequence

$\{f_{n_i}\}$ such that $n_1 \le n_2 \le n_3 \le \ldots$ etc

and $$\|f_{n_{i+1}} - f_{n_i}\|_p < \frac{1}{2^i} \qquad i = 1, 2, \ldots$$

Let $$g_k := \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}| \qquad g := \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|$$

By Minkowski's inequality, $\|g_k\|_p < 1$ for $k = 1, \ldots$

$\Rightarrow \int (g_k)^p \, d\mu < 1$ for all k. By Fatou's lemma,

$$\int \varliminf_{k\to\infty} (g_k)^p \, d\mu \le \varliminf_{k\to\infty} \int (g_k)^p \, d\mu < 1$$

and $\varliminf_{k\to\infty} (g_k)^p = (g)^p$

$\Rightarrow \|g\|_p < 1 \Rightarrow g(x) < \infty$ for almost all $x \in E$.

Thus the series $f_{n_1} + g$ converges absolutely for almost all $x \in E$.

$$f(x) := \begin{cases} f_{n_1}(x) + \sum_1^\infty (f_{n_{i+1}}(x) - f_{n_i}(x)) & \text{if the series conv.} \\ 0 & \text{otherwise (set of measure zero)} \end{cases}$$

Since $f_{n_1} + \sum_{i=1}^{k-1} f_{n_{i+1}} - f_{n_i} = f_{n_k}$, it follows that $f_{n_k} \to f$ almost everywhere.

Having found a pointwise limit almost everywhere for $\{f_{n_i}\}$, we now prove that f is the L^p limit of $\{f_n\}$.

Choose $\varepsilon > 0$. $\exists N$ so that $m, n \geq N \Rightarrow \|f_m - f_n\|_p < \varepsilon$.

For every $m > N$, by Fatou's lemma,

$$\int_E |f - f_m|^p d\mu \leq \varliminf_{i \to \infty} \int_E |f_{n_i} - f_m|^p d\mu \leq \varepsilon^p .$$

this proves $f - f_m \in L^p(\mu)$ and therefore $f \in L^p$ (since $f_m \in L^p$ and $f = f_m + (f - f_m)$)

Finally, since we've shown

$$\int_E |f - f_m|^p \le \varepsilon^p \quad \text{for } m \geq N, \text{ we}$$

conclude $\|f - f_m\|_p \to 0$ as $m \to \infty$ and done.

Case 2: $p = \infty$

Let $\{f_n\}$ be Cauchy in $L^\infty(\mu)$.

$$A_k := \{ x \mid |f_k(x)| > \|f_k\|_\infty \}$$

$$B_{m,n} := \{ x \mid |f_n(x) - f_m(x)| > \|f_m - f_n\|_\infty \}.$$

A_k and $B_{m,n}$ are sets of measure zero.

$$C = \left(\bigcup_{k=1}^\infty A_k \right) \cup \left(\bigcup_{m=1}^\infty \bigcup_{n=1}^\infty B_{m,n} \right)$$

then $\mu(C) = 0$.

On $E - C$ we have f_n converging uniformly to a bounded function f. (Via the completeness of $\mathbb{R}$ or $\mathbb{C}$.)

Extend the limit f to all of E by defining $f(x) = 0$ if $x \in C$. Then $f \in L^\infty(\mu)$ and $\|f_n - f\|_\infty \to 0$ as $n \to \infty$ by construction. //