Recall Lebesgue Dominated Convergence theorem:

LDCT: Suppose $\{f_n\}_1^\infty$ is a sequence of measurable functions on $(E, \mathcal{B}, \mu)$ and let f be a measurable function to which $\{f_n\}_1^\infty$ converges μ-almost everywhere. If there is a μ-integrable function g for which $|f_n| \leq g$ almost everywhere $\forall n \geq 1$ then f is integrable and

$$\lim_{n\to\infty} \int |f_n - f| \, d\mu \quad \text{and} \quad \lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

the theorem tells us that pointwise convergence (& control by g) implies that you can take the limit ($\lim_{n\to\infty}$) under the integral sign.

You definitely need this control... if you take a sequence of functions $\phi_n \geq 0$, $\int \phi_n = 1$, $\lim_{n\to\infty} \phi_n(x) = 0$ if $x \neq 0$ (recall $T_{\phi_n} \xrightarrow[n\to\infty]{} \delta_0$) then $\phi_n \to 0$ almost everywhere wrt Lebesgue measure, but

$$\lim_{n\to\infty} \int \phi_n \, dx = 1 \neq \int 0 \cdot dx = 0$$

Finding a function g that is a pointwise upperbound for ϕ_n can be a hassle. The control it provides can also be provided if you happen to know $\|f_n\|_{L^1(d\mu)} \to \|f\|_{L^1(d\mu)}$ as $n \to \infty$:

Theorem 3.3.5. Let $(E, \mathcal{B}, \mu)$ be a measure space, $\{f_n\}_1^\infty \cup \{f\} \subseteq L^1(\mu)$ and assume $f_n \to f$ almost everywhere wrt μ. Then

$$\lim_{n\to\infty} \Big| \|f_n\|_{L^1} - \|f\|_{L^1} - \|f_n - f\|_{L^1} \Big|$$

$$= \lim_{n\to\infty} \int \Big| |f_n| - |f| - |f_n - f| \Big| \, d\mu = 0$$

In particular, if $\|f_n\|_{L^1} \to \|f\|_{L^1}$ then $\|f_n - f\|_{L^1} \to 0$ and $f_n \to f$ almost everywhere in μ

proof: See book.

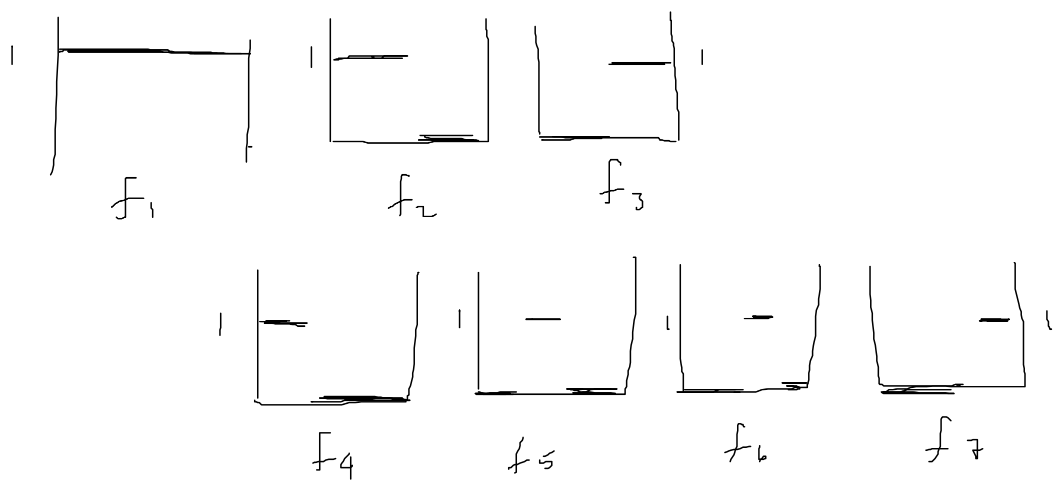
We now want to move on to the problem of proving that $L^1(\mu)$ is complete. We've given it a norm but now we need

$$\|f_n - f_m\|_{L^1} < \varepsilon \quad n, m \geq N_\varepsilon \implies \|f_n - f\|_{L^1} \to 0 \text{ for some } f \in L^1(\mu)$$

We see that we'll need something different from pointwise almost everywhere convergence from the following example:

Fix $[0,1]$ and define

$$f_{2^m+l} = 1_{[2^{-m}l,\, 2^{-m}(l+1)]} \qquad \begin{array}{l} m \geq 0 \\ 0 \leq l < 2^m \end{array}$$



then $\|f_n\|_{L^1(\mu)} = 2^{-m}$ if $2^{-m} \leq n < 2^{m+1}$ and

$\mu =$ lebesgue meas.

$\Rightarrow \|f_n - 0\|_{L^1(\mu)} \to 0$ as $n \to \infty$ but

$f_n \not\to 0$ almost everywhere wrt μ.

i.e. $\|f_n\|_{L^1(\mu)} \to 0 \not\Rightarrow f_n \to 0$ almost everywhere in μ.

So we see that convergence in L^1 is <u>weaker</u> than pointwise convergence almost everywhere.

Note: It is true that we can find a subsequence of $\{f_n\}$ so that the subsequence converges to 0 almost everywhere wrt μ. We will prove that this is true in general.

We now introduce a notion of convergence that is weaker than (pointwise) almost everywhere wrt. μ convergence. This convergence will be the one we use to prove $L^1(\mu)$ is complete.

defn: $\{f_n\}_1^\infty$ a sequence of measurable functions on $(E, \mathcal{B}, \mu)$ converges in μ-measure to a measurable function f if

$$\mu(|f_n - f| \geq \varepsilon) \to 0 \text{ as } n \to \infty \text{ for every } \varepsilon > 0$$

in which case we write $f_n \to f$ in μ-measure

Recall Markov's theorem which included

$$\lambda \mu(f \geq \lambda) \leq \int f \, d\mu \quad \text{(if } f \text{ non-neg } \mathcal{B} \text{ meas)}$$

$$\Rightarrow \quad \varepsilon \mu(|f_n - f| \geq \varepsilon) \leq \int |f_n - f| \, d\mu.$$

$\Rightarrow$ if $\|f_n - f\|_{L^1(\mu)} \to 0$ as $n \to \infty$ RHS $\to 0$ as $n \to \infty$

$\Rightarrow$ LHS $\to 0$ as $n \to \infty$ if $\varepsilon > 0 \Rightarrow$ $f_n \to f$ in μ measure

So we know that convergence in $L^1(\mu)$ implies convergence in μ-measure.

$\Rightarrow$ convergence in μ-measure is <u>weaker</u> than convergence almost everywhere wrt μ

(since our step function example on page 3 of these notes has $f_n \to 0$ in μ-measure but $f_n \not\to 0$ almost everywhere wrt μ.)

<u>Note</u>: if $f_n \to f$ and $f_n \to g$ in <u>μ-measure</u> then $f = g$ almost everywhere:

$$\mu(|f-g| \geq \varepsilon) \leq \mu(|f_n - f| \geq \varepsilon/2) + \mu(|f_n - g| \geq \varepsilon/2) \quad \circledast$$

and RHS $\to 0$ as $n \to \infty$ since

$f_n \to f$ in μ-measure

$f_n \to g$ in μ-measure

$\Rightarrow \mu(|f-g| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0 \Rightarrow f = g$ almost everywhere in μ

<u>Note</u>: $\circledast$ follows since

$$|f-g| \geq \varepsilon \Rightarrow |f - f_n| \geq \varepsilon/2 \text{ or } |g - f_n| \geq \varepsilon/2$$

$$\Rightarrow \{|f-g| \geq \varepsilon\} \subseteq \{|f-f_n| \geq \varepsilon/2\} \cup \{|g - f_n| \geq \varepsilon/2\}.$$

Why did

$$\mu(|f-g| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0$$
$$\Rightarrow \mu(f \neq g) = 0 \;?$$

because

$$\mu(f \neq g) = \lim_{\varepsilon \downarrow 0} \mu(|f-g| \geq \varepsilon) = \lim_{n \to \infty} \mu(|f-g| \geq \tfrac{1}{n}) = 0$$

$\Rightarrow$ convergence in μ-measure
convergence in $L^1(\mu)$
convergence almost everywhere wrt μ

$\rightarrow$ all determine the limit function f up to a set of measure zero.

We want to better understand how convergence in μ-measure relates to convergence almost everywhere wrt μ.

Theorem 3.3.7: Let $\{f_n\}_1^\infty$ be a sequence of $\mathbb{R}$-valued measurable functions on the measure space $(E, \mathcal{B}, \mu)$.

Then

1) $\exists$ $\mathbb{R}$-valued, measurable function f for which

$$(*) \quad \lim_{m \to \infty} \mu\left(\sup_{n \geq m} |f - f_n| \geq \varepsilon\right) = 0 \quad \forall \varepsilon > 0$$

$\Longleftrightarrow$

2) $$\lim_{m \to \infty} \mu\left(\sup_{n \geq m} |f_n - f_m| \geq \varepsilon\right) = 0 \quad \forall \varepsilon > 0$$

theorem 3.3.7 cont'd:

Moreover, $\circledast$ implies that $f_n \to f$ "almost everywhere wrt μ" as well as "in μ-measure".
Finally, if $\mu(E) < \infty$ then

$$\circledast \iff f_n \to f \text{ almost everywhere in } \mu.$$

Consider the sequence of characteristic functions on page 3 of these notes. Then for any m

$$\left\{ \sup_{n \geq m} |f_n - f_m| \geq \varepsilon \right\} = [0,1] \quad \text{if} \quad \varepsilon < 1$$

$$\Rightarrow \mu\left(\sup_{n \geq m} |f_n - f_m| \right) = \mu(E) \qquad \forall m$$

$$\Rightarrow \lim_{m \to \infty} \mu\left(\sup_{n \geq m} |f_n - f_m| \geq \varepsilon \right) \neq 0$$

so $\circledast$ doesn't hold.

Which is good because $\mu(E) < \infty$ and by the theorem, if $\circledast$ held then $f_n \to 0$ almost everywhere wrt μ

The theorem seems plausible because the quantity $\circledast$ has a pointwise feel to it.

proof of theorem 3.3.7:

2) $\Rightarrow$ 1):

Let $\Delta = \{x \in E \mid \lim_{n\to\infty} f_n(x)$ doesn't exist in $\mathbb{R}\}$.

for $m \geq 1$ and $\varepsilon > 0$, define

$$\Delta_m(\varepsilon) = \{x \in E \mid \sup_{n \geq m} |f_n(x) - f_m(x)| \geq \varepsilon\}.$$

then $$\Delta = \bigcup_{l=1}^{\infty} \bigcap_{m=1}^{\infty} \Delta_m(1/l).$$

[Why? assume $x \in \Delta$. then $\{f_n(x)\}$ is not a Cauchy sequence. $\Rightarrow \exists\, \varepsilon_0 > 0$ so that for any m $\exists\, n \geq m$ with $|f_n(x) - f_m(x)| \geq \varepsilon_0$. Pick $l_0 \ni 1/l_0 < \varepsilon_0$. Then for any m $\exists\, n \geq m \ni |f_n(x) - f_m(x)| \geq 1/l_0$

$$\Rightarrow x \in \Delta_m(1/l_0) \quad \forall m$$

$$\Rightarrow x \in \bigcap_1^{\infty} \Delta_m(1/l_0)$$

$$\Rightarrow x \in \bigcup_{l=1}^{\infty} \bigcap_{m=1}^{\infty} \Delta_m(1/l) \Rightarrow \Delta \subseteq \bigcup_1^{\infty} \bigcap_1^{\infty} \Delta_m(1/l).$$

Proving $\bigcup_1^{\infty} \bigcap_1^{\infty} \Delta_m(1/l) \subseteq \Delta$ is similar.]

Now, ⊛ is "$\lim_{m\to\infty} \mu(\Delta_m(\varepsilon)) = 0 \quad \forall \varepsilon > 0$"

$$\Rightarrow \mu\left(\bigcap_{m=1}^{\infty} \Delta_m(\varepsilon)\right) = \lim_{m\to\infty} \mu(\Delta_m(\varepsilon)) = 0 \qquad \forall \varepsilon > 0.$$

$$\Rightarrow \mu(\Delta) \le \sum_{l=1}^{\infty} \mu\left(\bigcap_{m=1}^{\infty} \Delta_m(1/l)\right) = 0.$$

$\Rightarrow \mu(\Delta) = 0 \Rightarrow f_n$ converges almost everywhere wrt μ.

let
$$f(x) := \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if } x \in \Delta \\ 0 & \text{if } x \notin \Delta \end{cases}$$

by lemma 3.3.1 f is a measurable function.
By construction, f is $\mathbb{R}$-valued. So we need to show

$$\lim_{m\to\infty} \mu\left(\sup_{n\ge m} |f-f_n| \ge \varepsilon\right) = 0 \qquad \forall \varepsilon > 0.$$

Since $|f_n - f| \le |f_n - f_m| + |f - f_m|$,

$$\sup_{n\ge m} |f_n - f| \le \sup_{n \ge m} \left(|f_n - f_m| + |f - f_m|\right)$$

$$\le \sup_{n\ge m} 2|f_n - f_m|$$ almost everywhere in μ (since $f_n \to f$ almost everywhere in μ)

$$\Rightarrow \left\{x \,\middle|\, \sup_{n\ge m} |f_n - f| \ge \varepsilon\right\} \subseteq \left\{x \,\middle|\, \sup_{n\ge m} |f_n - f_m| \ge \varepsilon/2\right\}.$$

$$\Rightarrow \mu\left(\sup_{n\ge m} |f_n - f| \ge \varepsilon\right) \le \mu\left(\sup_{n\ge m} |f_n - f_m| \ge \varepsilon/2\right)$$

by assumption,

$$\lim_{m\to\infty} \mu\left(\sup_{n\geq m} |f_n - f_m| \geq \varepsilon/2\right) = 0.$$

$\Rightarrow \lim_{m\to\infty} \mu\left(\sup_{n\geq m} |f_n - f| \geq \varepsilon\right) = 0.$ Since $\varepsilon > 0$ was arbitrary, this finishes proving 2) $\Rightarrow$ 1).

1) $\Rightarrow$ $f_n \to f$ μ-almost everywhere:

let $\Delta = \{x \in E \mid \lim_{n\to\infty} f_n(x) - f(x)$ does not exist in $\mathbb{R}\}$.

let $\Delta_m(\varepsilon) = \{x \in E \mid \sup_{n\geq m} |f_n(x) - f(x)| \geq \varepsilon\}$

Then $\Delta = \bigcup_{l=1}^{\infty} \bigcap_{m=1}^{\infty} \Delta_m(1/l)$.

Since 1) holds, $\lim_{m\to\infty} \mu(\Delta_m(1/l)) = 0 \quad \forall l$.

$\Rightarrow \mu(\Delta) = 0$ as before $\Rightarrow f_n \to f$ μ-almost everywhere.

1) $\Rightarrow$ $f_n \to f$ in μ-measure:

We want to show

$\mu(|f_m - f| \geq \varepsilon) \to 0$ as $m \to \infty$ for any $\varepsilon > 0$.

$$\{x \in E \mid |f_m(x) - f(x)| \geq \varepsilon\} \subseteq \{x \in E \mid \sup_{n\geq m} |f_n(x) - f(x)| \geq \varepsilon\}$$

$$\Rightarrow \mu(|f_m - f| \geq \varepsilon) \leq \mu\left(\sup_{n\geq m} |f_n - f| \geq \varepsilon\right)$$

RHS $\to 0$ as $m \to \infty$ by 1) $\Rightarrow$ LHS $\to 0$ $\Rightarrow$ $f_m \to f$ in μ-measure

1) $\Rightarrow$ 2) :

$$\{x \mid \sup_{n\geq m} |f_n - f| \geq \varepsilon\} \subseteq \{x \mid \sup_{n\geq m} |f_n - f| \geq \varepsilon/2\} \cup \{x \mid \sup_{n\geq m} |f - f_m| \geq \varepsilon/2\}$$

$$\Rightarrow \mu\left(\sup_{n\geq m} |f_n - f_m| \geq \varepsilon\right) \leq \mu\left(\sup_{n\geq m} |f_n - f| \geq \varepsilon/2\right) + \mu\left(\sup_{n\geq m} |f - f_m| \geq \varepsilon/2\right)$$

$$= \mu\left(\sup_{n\geq m} |f_n - f| \geq \varepsilon/2\right) + \mu\left(|f - f_m| \geq \varepsilon/2\right)$$

as $m \to \infty$ the first term $\to 0$ by assumption 1)
as $m \to \infty$ the second term on the RHS $\to 0$ since
$f_m \to f$ in μ-measure.

$\therefore$ LHS $\to 0$ as $m \to \infty$ and 2) holds.

$\mu(E) < \infty$ and $f_n \to f$ $(a.e.\ \mu)$ $\Rightarrow$ 1) :

Let $\Delta_m(\varepsilon) = \{x \in E \mid \sup_{n\geq m} |f_n(x) - f(x)| \geq \varepsilon\}$.

then $\Delta_1(\varepsilon) \supseteq \Delta_2(\varepsilon) \supseteq \Delta_3(\varepsilon) \supseteq \ldots$

$\mu(\Delta_1(\varepsilon)) \leq \mu(E) < \infty$ by assumption.

theorem 3.1.6 $\Rightarrow$ $\lim_{m\to\infty} \mu(\Delta_m(\varepsilon)) = \mu\left(\bigcap_{m=1}^{\infty} \Delta_m(\varepsilon)\right)$

and $f_n \to f$ $(a.e.\ \mu)$ implies $\mu\left(\bigcap_{m=1}^{\infty} \Delta_m(\varepsilon)\right) = 0 \Rightarrow$

$$\lim_{m\to\infty} \mu(\Delta_m(\varepsilon)) = 0$$

$\Rightarrow$ $\circledast$ holds and 1) is true. ✓ //

Note: this theorem tells us:

$f_n \to f$ almost everywhere wrt μ.

and $\mu(E) < \infty$

$\Rightarrow$ $f_n \to f$ in μ-measure

Note: the criterion in 2):

$$\lim_{m\to\infty} \mu\left(\sup_{n\geq m} |f_n - f_m| \geq \varepsilon\right)$$

is a Cauchy-criterion for "almost everywhere wrt μ" convergence.

Now we'll give a Cauchy criterion for convergence "in μ-measure" as well as the compactness result.

<u>Theorem 3.3.10</u>: Let $\{f_n\}_1^\infty$ be a sequence of $\mathbb{R}$-valued, measurable functions on a measure space $(E, \mathcal{B}, \mu)$. Then $\exists$ an $\mathbb{R}$-valued measurable f such that $f_n \to f$ in μ-measure if and only if

$$\lim_{m\to\infty} \sup_{n\geq m} \mu\left(|f_n - f_m| \geq \varepsilon\right) = 0 \qquad \varepsilon > 0.$$

Furthermore, if $f_n \to f$ in μ-measure then $\exists$ a subsequence $\{f_{n_j}\}_{j=1}^\infty$ such that

$$\lim_{i\to\infty} \mu\left(\sup_{j\geq i} |f - f_{n_j}| \geq \varepsilon\right) = 0 \qquad \forall \varepsilon > 0$$

and therefore $f_{n_j} \to f$ almost everywhere wrt μ.

<u>Note</u>: if we look at the f_n characteristic functions defined on page 3 of these notes, we see that

if $2^n \leq m < 2^{n+1}$ and μ = Lebesgue measure

then $\mu\left(|f_n - f_m| \geq \varepsilon\right) \leq 2 \cdot 2^{-m}$ if $\varepsilon < 1$

$$\Rightarrow \lim_{m\to\infty} \sup_{n\geq m} \mu\left(|f_n - f_m| \geq \varepsilon\right) \leq \lim_{m\to\infty} 2 \cdot 2^{-m} = 0$$

which is consistent with what we knew before: $f_n \to 0$ in μ-measure (before, I argued it from $\|f_n - 0\|_{L^1(\mu)} \to f_n \to 0$ in μ-meas.)

proof of theorem 3.3.10:

($\Rightarrow$):

fix $\varepsilon > 0$.

$$\mu(|f_m - f_n| \geq \varepsilon) \leq \mu(|f - f_n| \geq \varepsilon/2) + \mu(|f - f_m| \geq \varepsilon/2)$$

we've assumed $f_n \to f$ in μ-measure.

$$\therefore \lim_{n\to\infty} \mu(|f - f_n| \geq \varepsilon/2) = 0$$

$\therefore$ given $\delta > 0$ $\exists$ N_δ so that $n \geq N_\delta \Rightarrow \mu(|f - f_n| \geq \varepsilon/2) < \delta$.

$$\Rightarrow \sup_{m \geq N} \mu(|f_m - f| \geq \varepsilon/2) < \delta.$$

$$\Rightarrow \mu(|f_N - f| \geq \varepsilon/2) + \sup_{m\geq N} \mu(|f_m - f| \geq \varepsilon/2) < 2\delta$$

$$\Rightarrow \sup_{m \geq N} \mu(|f_m - f_N| \geq \varepsilon)$$

$$\leq \sup_{m\geq N} \mu(|f - f_N| \geq \varepsilon/2) + \mu(|f - f_m| \geq \varepsilon/2$$

$$= \mu(|f - f_N| \geq \varepsilon/2) + \sup_{m \geq N} \mu(|f - f_m| \geq \varepsilon/2) < 2\delta.$$

$\therefore \lim_{N\to\infty} \sup_{m\geq N} \mu(|f_m - f_N| \geq \varepsilon) = 0$, as desired ✓

$(\Leftarrow)$:

assume $\lim_{m\to\infty} \sup_{n\geq m} \mu(|f_n - f_m| \geq \varepsilon) = 0$.

we want to prove $\exists$ limiting function f and $f_n \to f$ in μ-measure.

Since $\lim_{m\to\infty} \sup_{n\geq m} \mu(|f_n - f_m| \geq \varepsilon) = 0$,

if we fix i, $\lim_{m\to\infty} \sup_{n\geq m} \mu(|f_n - f_m| \geq 2^{-i-1}) = 0$.

Specifically, $\exists\, n_i$ so that $m \geq n_i$ implies $\sup_{n\geq m} \mu(|f_n - f_m| \geq 2^{-i-1}) < 2^{-i-1}$

Specifically,

$$\sup_{n\geq n_i} \mu(|f_n - f_{n_i}| \geq 2^{-i-1}) < 2^{-i-1}$$

In this way, we construct a subsequence $\{f_{n_i}\}$ of $\{f_n\}_1^\infty$.

$$\Rightarrow \mu\left(\sup_{j\geq i} |f_{n_j} - f_{n_i}| > 2^{-i}\right) \leq \mu\left(\bigcup_{j\geq i} \{|f_{n_{j+1}} - f_{n_j}| \geq 2^{-j-1}\}\right)$$

$$\leq \sum_{j=i}^{\infty} \mu(|f_{n_{j+1}} - f_{n_j}| \geq 2^{-j-1})$$

$$\leq 2^{-i}$$

we can now apply theorem 3.3.7 to the subsequence $\{f_{n_i}\}$ since we know

$$\lim_{i\to\infty} \mu\left(\sup_{j\geq i} |f_{n_j} - f_{n_i}| \geq \varepsilon \right) = 0 \qquad \text{for any } \varepsilon > 0$$

(for a given $\varepsilon > 0$ choose N so that $2^{-N-1} < \varepsilon$. then $i \geq N$ will work.)

So we know $\exists$ a limiting function f such that $f_{n_j} \to f$ almost everywhere wrt μ. and $f_{n_i} \to f$ in μ-measure

We now have convergence on a subsequence. We want to prove that

$$f_n \to f \quad \text{in } \mu\text{-measure.}$$

$$\mu(|f_m - f| \geq \varepsilon) \leq \overline{\lim_{i\to\infty}}\, \mu(|f_m - f_{n_i}| \geq \varepsilon/2) + \overline{\lim_{i\to\infty}}\, \mu(|f_{n_i} - f| \geq \varepsilon/2).$$

Since $f_{n_i} \to f$ in μ-measure, the second term on the RHS $= 0$

the first term on the RHS is $\leq \sup_{n\geq m} \mu(|f_m - f_n| \geq \varepsilon/2)$.

$$\therefore\ \mu(|f_m - f| \geq \varepsilon) \leq \sup_{n\geq m} \mu(|f_m - f_n| \geq \varepsilon/2)$$

and RHS $\to 0$ as $m \to \infty$ by assumption.

this proves $f_m \to f$ in μ-measure, as desired.

Finally, assume $f_n \to f$ in μ-measure.

then $\lim_{m\to\infty} \sup_{n\geq m} \mu(|f_n - f_m| \geq \varepsilon) = 0$.

$\Rightarrow$ we can construct our subsequence f_{n_i} as above and we know $f_{n_i} \to g$ almost everywhere with respect to μ. Also, $f_{n_i} \to f$ in μ-measure. $\Rightarrow$ $g = f$ almost everywhere $\Rightarrow$ $f_{n_i} \to f$ almost everywhere wrt μ, as desired. //

to sum up

"$\|f_n - f\|_{L^1(\mu)} \to 0$" $\Rightarrow$ "$f_n \to f$ in μ measure"

$\Rightarrow$ "$f_{n_i} \to f$ almost everywhere wrt μ for some subsequence"

"$\mu(E) < \infty$ and $f_n \to f$ almost everywhere wrt μ"

$\Longrightarrow$ "$f_n \to f$ in μ-measure"

<u>Next:</u> $L^1(\mu)$ is complete

continuous functions are dense in $L^1(|\cdot|)$ where $|\cdot|$ is the Lebesgue measure on $\mathbb{R}^n$.