Note: the correct definition of $\hat{\mathbb{R}}^2$ is

$$\overline{\mathbb{R}}^2 - \{(-\infty,\infty),(\infty,-\infty)\}.$$

since $+: \hat{\mathbb{R}}^2 \to \overline{\mathbb{R}}$ is continuous where

$$\alpha \pm \infty = \pm\infty + \alpha = \pm\infty \quad \text{as long as } \alpha \neq \mp\infty,$$

this is a mistake in the book (thanks Ari!)

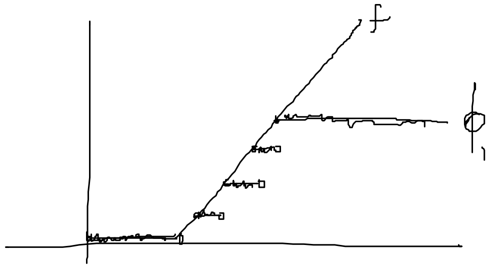
Okay, we've defined the Lebesgue integral for nonnegative simple measurable functions. We now use them to define the Lebesgue integral for nonnegative measurable functions.

Given f, and n, we define a simple function

$\phi_n : E \to [0,\infty)$ by

$$\phi_n = \sum_{k=0}^{4^n-1} \frac{k}{2^n} 1_{f \in [\frac{k}{2^n}, \frac{k+1}{2^n})} + 2^n 1_{f \geq 2^n}$$

e.g. $f : [0,\infty) \to [0,\infty)$ $\quad f(x) = [x-1]_+$



$n=1$

$$\phi_1(x) = \begin{cases} 0 & x \in [0, 3/2) \\ 1/2 & x \in [3/2, 2) \\ 1 & x \in [2, 5/2) \\ 3/2 & x \in [5/2, 3) \\ 2 & x \in [3, \infty) \end{cases}$$

So ϕ_n is a simple function with 4^n values.
ϕ_n is bounded.

If f is a bounded function then ϕ_n converges uniformly to f in the metric of $\mathbb{R}$.

If f is unbounded then ϕ_n converges uniformly to f in the metric of $\overline{\mathbb{R}}$

Either way, ϕ_n converges uniformly to f and since

$\int \phi_n(x)\, d\mu(x)$ is defined (since ϕ_n simple & measurable)

we might hope

$$\lim_{n\to\infty} \int \phi_n(x)\, d\mu(x) = \int \lim_{n\to\infty} \phi_n(x)\, d\mu(x) = \int f(x)\, d\mu(x)$$

i.e. <u>define</u>

$$\int f(x)\, d\mu(x) := \lim_{n\to\infty} \left[\sum_{k=0}^{4^n-1} \frac{k}{2^n} \mu\left(f \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right) + 2^n \mu(f \geq 2^n) \right]$$

we define this to be the Lebesgue integral of a nonnegative measurable function f.

Q1: What if we had approximated f in some other way? Could we get a different answer for $\int f(x)\,d\mu(x)$??

Q2: Does the definition even agree for f a simple function (nonnegative, measurable)

lemma 3.2.6. Let $(E, \mathcal{B}, \mu)$ be a measure space and suppose $\{\phi_n\}_1^\infty$ and ψ are nonnegative measurable simple functions on $(E, \mathcal{B})$. If $\phi_n \le \phi_{n+1}$ for all $n \ge 1$ and $\psi \le \lim_{n\to\infty} \phi_n$ then $\int \psi\,d\mu \le \lim_{n\to\infty} \int \phi_n\,d\mu$. In particular, for any nonnegative measurable function f and any nondecreasing sequence $\{\psi_n\}$ of nonnegative simple measurable functions ψ_n which tend to f as $n\to\infty$, ($\psi_n \uparrow f$)

$$\lim_{n\to\infty} \int \psi_n\,d\mu = \lim_{n\to\infty} \int \phi_n\,d\mu$$

where ϕ_n is defined as before:

$$\phi_n = \sum_{k=0}^{4^n-1} \frac{k}{2^n} \mathbb{1}_{f \in [k/2^n, \frac{k+1}{2^n})} + 2^n \mathbb{1}_{f \ge 2^n}$$

<u>Note</u> we say nothing about

$$\int f\,d\mu < \infty. \quad \text{it could} = \infty.$$

<u>Note</u>: the second part of the theorem implies that the "new" definition of Lebesgue integral is the same as the first one if f is a simple nonnegative measurable function. Since in such a case, we could take $\{\psi_n\} = \{f\}_1^\infty$ and this is an alternate sequence of functions that converges to f as $n \to \infty$. ($\psi_n \uparrow f$.)

<u>Proof of lemma</u>: First we prove the second part, assuming the first part. Let ϕ_n be our "standard" simple approximating functions. Let ψ_n be another sequence $\ni \psi_n \uparrow f$.

$$\Rightarrow \quad \phi_m \le \lim_{n\to\infty} \psi_n = f \text{ for any } m$$

$$\psi_m \le \lim_{n\to\infty} \phi_n = f \text{ for any } m$$

$\Rightarrow$ by part 1,

$$\int \phi_m\,d\mu \le \lim_{n\to\infty} \int \psi_n\,d\mu \quad \text{for each } m$$

$$\Rightarrow \lim_{m\to\infty} \int \phi_m\,d\mu \le \lim_{n\to\infty} \int \psi_n\,d\mu$$

Similarly, $\lim_{m\to\infty} \int \psi_m\,d\mu \le \lim_{n\to\infty} \int \phi_n\,d\mu \Rightarrow$ the limits are equal.

So it suffices to prove the first part.

Case 1: $\mu(\psi = \infty) > 0$ (expect integrals $\uparrow \infty$ as $n \to \infty$)

Case 2: $\mu(\psi > 0) = \infty$ (expect integrals $\uparrow \infty$ since ψ simple)

Case 3: $\mu(\psi > 0) < \infty$ and $\mu(\psi = \infty) = 0$ (finite integrals possible.)

Case 1: assume $\mu(\psi = \infty) > 0$. Fix $M < \infty$.

since $\phi_n \le \phi_{n+1} \le \cdots$

$$\{\phi_n \ge M\} \subseteq \{\phi_{n+1} \ge M\} \subseteq \text{etc.}$$

since $\psi \le \lim\limits_{n\to\infty} \phi_n$

$$\Rightarrow \mu(\phi_n \ge M) \uparrow \mu\left(\bigcup_{n=1}^{\infty} \{\phi_n \ge M\}\right) \ge \mu(\psi \ge M)$$

$$\ge \mu(\psi = \infty) = \varepsilon$$

$$\Rightarrow \lim_{n\to\infty} \int \phi_n \, d\mu \ge \lim_{n\to\infty} M\mu(\phi_n \ge M) \ge M\varepsilon.$$

$\Rightarrow$ since M was arbitrary, we see

$$\lim_{n\to\infty} \int \phi_n \, d\mu = \infty$$

and $$\int \psi \, d\mu = \sum_{\alpha \in \text{range}\{\psi\}} \alpha \mu(\psi = \alpha) \ge \infty \cdot \varepsilon = \infty.$$

$$\Rightarrow \int \psi \, d\mu \le \lim_{n\to\infty} \int \phi_n \, d\mu. \quad \checkmark$$

Case 2: $\mu(\psi > 0) = \infty$.

Since ψ is simple, $\exists\, \varepsilon > 0 \ni \varepsilon < \psi$. $\left(\varepsilon < \min\{\alpha \in \text{range}\,\psi \mid \alpha > 0\}\right)$

$$\Rightarrow \int \psi \, d\mu \geq \min\{\alpha \in \text{range}\,\psi \mid \alpha > 0\} \cdot \mu(\psi > 0) = \infty.$$

Since $\psi \leq \lim_{n \to \infty} \phi_n$, $\exists\, N \ni n \geq N \Rightarrow \phi_n > \varepsilon$.

$$\Rightarrow \mu(\phi_n > \varepsilon) \uparrow \mu\left(\bigcup_1^\infty \{\phi_n > \varepsilon\}\right) \geq \mu(\{\psi > 0\}) = \infty.$$

and $\phi_n \geq \varepsilon 1_{\{\phi_n \geq \varepsilon\}}$

$$\Rightarrow \int \phi_n \, d\mu \geq \varepsilon \cdot \mu(\phi_n \geq \varepsilon) \geq \varepsilon \mu(\phi_n > \varepsilon)$$

$$\Rightarrow \lim_{n \to \infty} \int \phi_n \, d\mu \geq \lim_{n \to \infty} \varepsilon \mu(\phi_n > \varepsilon) = \infty \geq \int \psi \, d\mu. \checkmark$$

Case 3: $\mu(\psi > 0) < \infty$ and $\mu(\psi = \infty) = 0$

define $\hat{E} = \{0 < \psi < \infty\}$. $\mu(\hat{E}) < \infty$ by assumption.

Further, $\psi = \psi 1_{\hat{E}}$ everywhere except on a set of measure zero (where $\psi = \infty$)

and $\psi 1_{\hat{E}}$ is a characteristic function.

$$\rightarrow \int \psi \, d\mu = \int \psi 1_{\hat{E}} \, d\mu = \int_{\hat{E}} \psi \, d\mu$$

and $\phi_n \geq \phi_n 1_{\hat{E}} \Rightarrow \int \phi_n \, d\mu \geq \int \phi_n 1_{\hat{E}} \, d\mu = \int_{\hat{E}} \phi_n \, d\mu$

So it suffices to prove

$$\int_{\hat{E}} \psi \, d\mu \leq \lim_{n\to\infty} \int_{\hat{E}} \phi_n \, d\mu$$

Since ψ is simple and finite on $\hat{E}$, $\exists \varepsilon > 0$ and $M < \infty$ so that

$$\varepsilon \leq \psi \leq M$$

fix $\delta \in (0, \varepsilon)$ $\quad E_n := \{ x \mid \phi_n(x) \geq \psi(x) - \delta \}$

E_n is measurable. (why?)

since $\phi_n \leq \phi_{n+1} \leq \ldots$ we have $E_n \uparrow \hat{E}$ $\Rightarrow$

$$\lim_{n\to\infty} \int_{\hat{E}} \phi_n \, d\mu \geq \lim_{n\to\infty} \int_{E_n} \phi_n \, d\mu \geq \lim_{n\to\infty} \int_{E_n} \psi \, d\mu - \delta\mu(E_n)$$

$$\geq \lim_{n\to\infty} \int_{E_n} \psi \, d\mu - \delta\mu(\hat{E})$$

$$= \lim_{n\to\infty} \int_{\hat{E}} \psi \, d\mu - \int_{\hat{E}-E_n} \psi \, d\mu - \delta\mu(\hat{E})$$

$$\geq \lim_{n\to\infty} \int_{\hat{E}} \psi \, d\mu - M\mu(\hat{E}-E_n) - \delta\mu(\hat{E})$$

$$= \int_{\hat{E}} \psi \, d\mu - \delta\mu(\hat{E})$$

taking $\delta \downarrow 0$ $\quad \int_{\hat{E}} \psi \, d\mu \leq \lim_{n\to\infty} \int \phi_n \, d\mu$ as desired. //

Now we're done defining the Lebesgue integral of a nonnegative $\overline{\mathbb{R}}$-valued measurable function.

Lemma 3.27 Let f and g be nonnegative measurable functions on the measure space $(E, \mathcal{B}, \mu)$ and $\alpha, \beta \in [0, \infty]$ then

$$\int \alpha f + \beta g \, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu$$

and if $f \le g$ then $\int f \, d\mu \le \int g \, d\mu$

and if $\int f \, d\mu < \infty$ then $f \le g \Rightarrow \int g - f \, d\mu = \int g \, d\mu - \int f \, d\mu$

proof: let ϕ_n be simple fxns $\ni \phi_n \uparrow f$

let $\tilde{\phi}_n$ be simple fxns $\ni \tilde{\phi}_n \uparrow g$.

$\rightarrow \alpha \phi_n + \beta \tilde{\phi}_n = \psi_n$ are simple fxns $\ni \psi_n \uparrow \alpha f + \beta g$.

$\Rightarrow \int \alpha f + \beta g \, d\mu = \lim_{n \to \infty} \int \psi_n \, d\mu = \lim_{n \to \infty} \alpha \int \phi_n \, d\mu + \beta \int \tilde{\phi}_n \, d\mu = \alpha \int f + \beta \int g$

if $f \le g$ then since $\phi_m \uparrow f$, $\tilde{\phi}_n \uparrow g$

for each m, $\phi_m \le \lim_{n\to\infty} \tilde{\phi}_n = g$

$$\to \int \phi_m \, d\mu \le \lim_{n\to\infty} \int \tilde{\phi}_n \, d\mu \qquad \text{(valid since } \phi_m, \tilde{\phi}_n \text{ step)}$$

$$= \int g \, d\mu \qquad \text{by lemma 3.2.6}$$

$$\to \lim_{m\to\infty} \int \phi_m \, d\mu = \int f \, d\mu \le \int g \, d\mu.$$

etc. //

<u>Theorem 3.2.8</u> (Markov's inequality) If f is a nonnegative measurable function on $(E, \mathcal{B}, \mu)$ then

$$\lambda \mu(f \ge \lambda) \le \int_{\{f \ge \lambda\}} f \, d\mu \le \int f \, d\mu \qquad \lambda > 0$$

In particular, $\int f \, d\mu = 0 \iff \mu(f > 0) = 0$

and $\int f \, d\mu < \infty \Rightarrow \mu(f = \infty) = 0$.

<u>proof:</u> the inequality is trivial since

$$\lambda 1_{f \ge \lambda} \le f 1_{f \ge \lambda} \le f$$

and ~~by~~ all three are nonnegative measurable functions so by lemma 3.2.7,

$$\int \lambda 1_{f \ge \lambda} \, d\mu \le \int f 1_{f \ge \lambda} \, d\mu \le \int f \, d\mu \quad \checkmark$$

if $\int f \, d\mu = 0$ then $\lambda \mu(f \ge \lambda) = 0 \ \forall \lambda \Rightarrow \mu(f \ge \lambda) = 0 \ \forall \lambda$

$$\Rightarrow \lim_{\lambda \downarrow 0} \mu(f \ge \lambda) = 0 \Rightarrow \mu(f > 0) = 0$$

if $\mu(f>0)=0$ then since

$$0 \le f \le 0 \cdot 1_{f=0} + \infty \cdot 1_{f>0}$$

$$0 \le \int f \, d\mu \le 0 \cdot \mu(f=0) + \infty \cdot \mu(f>0) = 0 \Rightarrow \int f \, d\mu = 0.$$

Now show $\int f \, d\mu < \infty \Rightarrow \mu(f=\infty)=0$

Since $\lambda \mu(f \ge \lambda) \le \int f \, d\mu \quad \forall \lambda > 0$,

$$\mu(f \ge \lambda) \le \frac{1}{\lambda} \int f \, d\mu$$

take $\lambda \uparrow \infty$. LHS: $\mu(f \ge \lambda) \downarrow$ and $\lim_{\lambda \uparrow \infty} \mu(f \ge \lambda) \ge \mu(f=\infty)$

RHS: $\frac{1}{\lambda} \int f \, d\mu \downarrow 0$.

$\rightarrow \mu(f=\infty) = 0$ as desired. //

What if $f \ge 0$ isn't true? We know that $f_+ = \max\{f, 0\}$ is a measurable function as is $f_- = -\min\{f, 0\}$.

$|f| = f_+ + f_-$ and $f = f_+ - f_-$

We say $\int f \, d\mu$ exists if $\int f^+ d\mu \ne \infty$ or $\int f^- d\mu \ne \infty$.

(we want to avoid $\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu = \infty - \infty$.)

So if $f : E \to \overline{\mathbb{R}}$ then

$$\int f \, d\mu := \int f^+ d\mu - \int f^- d\mu$$ as long as the difference makes sense

<u>Note:</u> $\int f\,d\mu \in \overline{\mathbb{R}}$.

if $\Gamma \in \mathcal{B}$ then $\int_\Gamma f\,d\mu$ exists too. Why?

$$\int_\Gamma f^+ d\mu = \int f^+ 1_\Gamma \qquad \int_\Gamma f^- d\mu = \int f^- 1_\Gamma$$

$\to \int_\Gamma f^+ \leq \int f^+$ and $\int_\Gamma f^- \leq \int f^- d\mu$.

$\Rightarrow$ if no more than one of $\int f^+ d\mu$ and $\int f^- d\mu$ is ∞ then no more than one of $\int_\Gamma f^+ d\mu$ & $\int_\Gamma f^- d\mu$ is ∞

$$\to \int_\Gamma f\,d\mu = \int_\Gamma f^+ d\mu - \int_\Gamma f^- d\mu \quad ✓$$

Similarly, if $\Gamma_1 \cap \Gamma_2 = \phi$ then $\int_{\Gamma_1 \cup \Gamma_2} f\,d\mu = \int_{\Gamma_1} f\,d\mu + \int_{\Gamma_2} f\,d\mu$

and
$$\left| \int f\,d\mu \right| = \left| \int f^+ d\mu - \int f^- d\mu \right|$$
$$\leq \left| \int f^+ d\mu \right| + \left| \int f^- d\mu \right| = \int f^+ d\mu + \int f^- d\mu$$
$$= \int |f|\,d\mu \quad ✓$$

similarly, $\left|\int_\Gamma f\,d\mu\right| \le \int_\Gamma |f|\,d\mu$

$\Rightarrow \mu(\Gamma)=0 \Rightarrow \int_\Gamma f\,d\mu = 0$ for any measurable function f.

<u>Q:</u> Do we still know that $\int \alpha f+\beta g\,d\mu = \alpha\int f\,d\mu + \beta\int g\,d\mu$?

we know this for <u>nonnegative</u> measurable functions

<u>Q:</u> Do we still know that $f\le g \Rightarrow \int f\,d\mu \le \int g\,d\mu$?

$f\le g \Rightarrow f^+\le g^+$ and $g^-\le f^-$

$\Rightarrow \int f^+ d\mu \le \int g^+ d\mu$ & $\int g^- d\mu \le \int f^- d\mu$ (lemma 3.2.7)

$\rightarrow$ $\Rightarrow -\int f^- d\mu \le -\int g^- d\mu$

$\Rightarrow \int f^+ d\mu - \int f^- d\mu \le \int g^+ d\mu - \int g^- d\mu$

$\Rightarrow \int f\,d\mu \le \int g\,d\mu.$

for the linearity, it's a little harder, but it's true.

Lemma 3.2.11.

Let f and g be measurable functions for which

1) $\int f\,d\mu$ and $\int g\,d\mu$ exist

2) $\left(\int f\,d\mu, \int g\,d\mu\right) \in \hat{\mathbb{R}}^2$

then $\mu(f \otimes g \notin \hat{\mathbb{R}}^2) = 0$ and

$$\int_{f \otimes g \in \hat{\mathbb{R}}^2} f+g\,d\mu \quad \text{exists} \quad \underline{\text{and}}$$

$$\int_{f \otimes g \in \hat{\mathbb{R}}^2} f+g\,d\mu = \int f\,d\mu + \int g\,d\mu$$

Proof: see book.

Note: this is the most we have a right to since if $f(x_0) = \infty$ and $g(x_0) = -\infty$ at some $x_0 \in E$ then $f(x_0)+g(x_0)$ is not defined $\Rightarrow$ we shouldn't include such an x_0 in the domain of integration.

Given a measurable function f on $(E, \mathcal{B}, \mu)$ we define $\|f\|_{L^1(\mu)} := \int |f|\,d\mu$

and

<u>defn.</u> $f : E \to \overline{\mathbb{R}}$ a measurable function is <u>μ-integrable</u> if $\|f\|_{L^1(\mu)} < \infty$.

<u>defn:</u> $L^1(\mu) := L^1(E, \mathcal{B}, \mu) = \{$measurable $\overline{\mathbb{R}}$-valued fxns with $\|f\|_{L^1(\mu)} < \infty$.$\}$

<u>Note:</u> if $f \in L^1(\mu)$ then $\mu(f = \infty) = 0$ & $\mu(f = -\infty) = 0$

why? $\int |f| \, d\mu = \int f^+ d\mu + \int f^- d\mu < \infty$

$\Rightarrow \int f^+ d\mu < \infty \Rightarrow \mu(f^+ = \infty) = 0$ by Markov

$\int f^- d\mu < \infty \Rightarrow \mu(f^- = \infty) = 0$ by Markov

$\Rightarrow \mu(f = -\infty) = 0$

$\therefore \int f \, d\mu = \int f \mathbb{1}_{|f| \neq \infty} \, d\mu$

Further, $\| f - f\mathbb{1}_{|f| \neq \infty} \|_{L^1} = \int |f - f\mathbb{1}_{|f| \neq \infty}| = \infty \mu_{f=\infty} + \infty \mu(f = -\infty)$

$= 0 + 0 = 0$

$\Rightarrow$ f and $f\mathbb{1}_{|f| \neq \infty}$ are indistinguisable except on a set of 0 measure.

So without loss of generality, we'll assume $f \in L^1 \Rightarrow f$ is $\mathbb{R}$-valued, not $\overline{\mathbb{R}}$ valued.

Good news, we can easily add elts of L^1 now since we don't worry about

$$f(x_0) + g(x_0) = \infty - \infty \quad \text{for example}$$

Lemma 3.2.12 Let $(E, \mathcal{B}, \mu)$ be a measure space then $L^1(\mu)$ is a vector space and

$$\|\alpha f + \beta g\|_{L^1} \leq |\alpha| \, \|f\| + |\beta| \, \|g\|$$

for $\alpha, \beta \in \mathbb{R}$, $f, g \in L^1(\mu)$.

proof: $|\alpha f + \beta g| \leq |\alpha| \, |f| + |\beta| \, |g|$

and $|\alpha||f|$ is a nonneg meas. function as is $|\beta||g|$. $\Rightarrow$

$$\int |\alpha f + \beta g| \leq |\alpha| \int |f| \, d\mu + |\beta| \int |g| \, d\mu \text{ by}$$

lemma 3.2.7.

Corr: $\|f+g\| \leq \|f\| + \|g\|$ and $\| \; \|_{L^1}$ is a metric... if we know $\|f\|_{L^1} = 0 \Rightarrow f \equiv 0$. We don't know this so we introduce an equivalence relation.

$$f \sim g \quad \text{if} \quad \mu(g \neq f) = 0.$$

1) check this is an equivalence relation
2) check that $\|f\|_{L^1} = 0 \Rightarrow f \sim 0$

$\Rightarrow$ $L^1(\mu)/\sim$ is a normed vector space. We won't carry around the equivalence class notation

<u>Goal:</u> prove $L^1(\mu)$ is <u>complete</u>

<u>Lemma 3.3.1</u> Let $(E, \mathcal{B})$ be a measurable space and $\{f_n\}$ a sequence of measurable functions on $(E, \mathcal{B})$. Then

$$\sup_{n \geq 1} f_n \quad , \quad \inf_{n \geq 1} f_n \quad , \quad \overline{\lim_{n \to \infty}} f_n \quad , \quad \underline{\lim_{n \to \infty}} f_n$$

are all measurable. Furthermore,

$$\Delta := \{x \in E \mid \lim_{n \to \infty} f_n(x) \text{ exists}\} \in \mathcal{B}$$

and $f(x) = \begin{cases} 0 & x \notin \Delta \\ \lim_{n \to \infty} f_n(x) & x \in \Delta \end{cases}$ is a measurable fxn.

Proof:

- First assume $f_1 \le f_2 \le \dots$ $\{f_n\}$ is nondecreasing.

  then $\lim_{n\to\infty} f_n(x)$ exists for each $x \in E$. $(\Delta = E)$.

  is $\lim_{n\to\infty} f_n$ measurable? It suffices to check $\{x \mid \lim_{n\to\infty} f_n(x) > a\}$ is measurable $\forall\, a \in \mathbb{R}$

  Since f_n is nondecreasing,

  $$\{x \mid \lim_{n\to\infty} f_n(x) > a\} = \bigcup_n \{x \mid f_n > a\} \in \mathcal{B} \checkmark$$

  $\Rightarrow \lim_{n\to\infty} f_n$ is a measurable fxn on E

- Similarly, if f_n is nonincreasing, $\lim_{n\to\infty} f_n$ is a measurable function on E.

- Let $\{f_n\}$ be an arbitrary sequence of measurable functions. then

  $$g_n := f_1 \vee f_2 \vee \dots \vee f_n \text{ is measurable}$$

  and $g_1 \le g_2 \le \dots \Rightarrow \lim_{n\to\infty} g_n$ is measurable.

  but $\lim_{n\to\infty} g_n = \sup_n f_n \Rightarrow \sup_n f_n$ is measurable.

  Similarly, $\inf_n f_n$ is measurable.

Now by the same logic, if we fix

$g_m := \inf_{n \geq m} f_n$ is a measurable fxn.

and $g_1 \leq g_2 \leq g_3 \leq \ldots$

$\Rightarrow \lim_{n \to \infty} g_n$ is a measurable fxn

but $\lim_{n \to \infty} \inf_{n \geq m} f_n := \underline{\lim}_{n \to \infty} f_n \Rightarrow \underline{\lim}_{n \to \infty} f_n$ is measurable.

Similarly, $\overline{\lim}_{n \to \infty} f_n$ is a measurable function.

Finally, given two measurable functions g & h,

$\{x \mid g(x) = h(x)\}$ is in $\mathcal{B}$

$\Rightarrow \{x \mid \underline{\lim} f_n(x) = \overline{\lim} f_n(x)\} \in \mathcal{B}$

$\Rightarrow \{x \mid \lim_{n \to \infty} f_n(x) \text{ exists}\} \in \mathcal{B}$

$$\Rightarrow f(x) = \begin{cases} 0 & x \notin \mathcal{B} \\ \lim_{n \to \infty} f_n(x) & x \in \mathcal{B} \end{cases}$$ is measurable.

//