# Measure Spaces

We constructed $\overline{\mathcal{B}_{\mathbb{R}^N}}$ and $|\cdot|$ (Lebesgue measure). Before continuing on to integration, we'll study general measure spaces and then define integration in that context.

Given a set $E$, define $\mathcal{P}(E)$ to be the power set of $E$.

$$\mathcal{P}(E) = \{\Gamma \mid \Gamma \subseteq E\}.$$

an <u>algebra</u> over $E$ is a subset $\mathcal{A} \subseteq \mathcal{P}(E)$ that satisfies

a) $\emptyset \in \mathcal{A}$

b) $\Gamma \in \mathcal{A} \Rightarrow \Gamma^c \in \mathcal{A}$

c) $\Gamma_1, \Gamma_2 \in \mathcal{A} \Rightarrow \Gamma_1 \cup \Gamma_2 \in \mathcal{A}$

this then implies that an algebra is closed under finite unions and intersections.

A <u>$\sigma$-algebra</u> is an algebra that is closed under countable unions.

$\mathcal{A} = \{\emptyset, E\}$ is the smallest (sigma) algebra over $E$

$\mathcal{A} = \mathcal{P}(E)$ is the largest (sigma) algebra over $E$.

$\mathcal{A} = \overline{\mathcal{B}_{\mathbb{R}^N}}$ is a $\sigma$-algebra over $\mathbb{R}^N$

Lemma 3.1.2

The intersection of any collection of algebras (or sigma algebras) is again an algebra (or $\sigma$-algebra). In particular, given any nonempty subset $\mathcal{C} \subseteq \mathcal{P}(E)$ there is a unique minimal algebra $\mathcal{A}(E;\mathcal{C})$ and a unique minimal sigma algebra $\sigma(E;\mathcal{C})$ over $E$ containing $\mathcal{C}$.

proof: see book.

We call $\sigma(E;\mathcal{C})$ the sigma algebra generated by $\mathcal{C}$

if $E$ is a topological space and $\mathcal{G} \subseteq \mathcal{P}(E)$ is the class of open sets in $E$ then $\mathcal{B}_E := \sigma(E,\mathcal{G})$ is called the Borel $\sigma$-algebra

Given a set $E$ and a $\sigma$-algebra $\mathcal{B}$ over $E$ we call $(E,\mathcal{B})$ a measurable space.

if $(E,\mathcal{B})$ is a measurable space then the map $\mu : \mathcal{B} \to [0,\infty]$ is a measure on $(E,\mathcal{B})$ if

1) $\mu(\emptyset) = 0$

2) $\mu\left(\bigcup_1^\infty \Gamma_k\right) = \sum_1^\infty \mu(\Gamma_k)$ if $\Gamma_k \cap \Gamma_l = \emptyset$ $\forall k \neq l$.

$(E,\mathcal{B},\mu)$ is called a measure space if $\mu(E) < \infty$ then $(E, \mathcal{B}, \mu)$ is a <u>finite measure space</u>

if $\mu(E) = 1$ then $(E, \mathcal{B}, \mu)$ is a <u>probability space</u>

Given a measure space $(E, \mathcal{B}, \mu)$ one can always extend $\mu$ as a measure $\bar{\mu}$ on the $\sigma$-algebra $\overline{\mathcal{B}}^{\mu}$ of the sets $\Gamma \subseteq E$ with the property that $\exists$ $A, B \in \mathcal{B}$ with $A \subseteq \Gamma \subseteq B$ and $\mu(B - A) = 0$.
In this case, we define $\bar{\mu}(\Gamma) := \mu(A)$.
The $\sigma$-algebra $\overline{\mathcal{B}}^{\mu}$ is the <u>completion of $\mathcal{B}$ with respect to $\mu$</u>.

(This is what our $\overline{\mathcal{B}_{\mathbb{R}^N}}$ is. We first take the open sets in $\mathbb{R}^N$ and define $\mathcal{B}_{\mathbb{R}^N}$. We then define Lebesgue measure on $\mathcal{B}_{\mathbb{R}^N}$. And then we completed $\mathcal{B}_{\mathbb{R}^N}$ with respect to Lebesgue measure, resulting in $\overline{\mathcal{B}_{\mathbb{R}^N}}^{\lambda}$. Which we refer to as $\overline{\mathcal{B}_{\mathbb{R}^N}}$ with the Lebesgue measure being implicitly what the completion was taken with respect to.)

Just as you could define a relative topology by intersecting your open sets with a fixed set, you can define a measure space via restriction.

Let $(E, \mathcal{B}, \mu)$ be a measure space. Fix $E' \in \mathcal{B}$.

$$\mathcal{B}[E'] := \{ \Gamma \cap E' \mid \Gamma \in \mathcal{B} \} \text{ is a } \sigma\text{-algebra}$$

$$\mu\big|_{\mathcal{B}[E']} (\Gamma \cap E) := \mu(\Gamma \cap E')$$

then $\left(E', \mathcal{B}[E'], \mu\big|_{\mathcal{B}[E']}\right)$ is a measure space.

<u>Theorem 3.1.6</u> Let $(E, \mathcal{B}, \mu)$ be a measure space.

- If $\Gamma_1, \Gamma_2 \in \mathcal{B}$ and $\Gamma_1 \subseteq \Gamma_2$ then $\mu(\Gamma_1) \leq \mu(\Gamma_2)$.
- if $\mu(\Gamma_1) < \infty$ then $\mu(\Gamma_2 - \Gamma_1) = \mu(\Gamma_2) - \mu(\Gamma_1)$
- for $\{\Gamma_k\}_1^\infty \subset \mathcal{B}$ $\quad \Gamma_1 \subset \Gamma_2 \subset \dots \subset \Gamma_k \subset \Gamma_{k+1}$,

$$\mu(\Gamma_k) \uparrow \mu\left(\bigcup_1^\infty \Gamma_k\right) \text{ as } k \to \infty$$

- for $\{\Gamma_k\}_1^\infty \subset \mathcal{B}$ $\quad \Gamma_1 \supset \Gamma_2 \supset \dots \supset \Gamma_{k-1} \supset \Gamma_k \supset \dots$

  if $\mu(\Gamma_1) < \infty$ then $\mu(\Gamma_k) \downarrow \mu\left(\bigcap_1^\infty \Gamma_k\right)$ as $k \to \infty$

- $\mu\left(\bigcup_1^\infty \Gamma_k\right) \leq \sum_1^\infty \mu(\Gamma_k)$
- $\mu\left(\bigcup_1^\infty \Gamma_k\right) = \sum_1^\infty \mu(\Gamma_k)$ if $\mu(\Gamma_k \cap \Gamma_\ell) = \phi \quad \forall k \neq \ell$.

proof:

- Since $\Gamma_1 \subset \Gamma_2$, $\Gamma_2 = \Gamma_1 \cup (\Gamma_2 - \Gamma_1)$. $\Rightarrow \mu(\Gamma_2) = \mu(\Gamma_1) + \mu(\Gamma_2 - \Gamma_1)$
because $\mu(\Gamma_2 - \Gamma_1) \geq 0$ we have $\mu(\Gamma_2) \geq \mu(\Gamma_1)$,
and if $\mu(\Gamma_1) < \infty$ we have $\mu(\Gamma_2 - \Gamma_1) = \mu(\Gamma_2) - \mu(\Gamma_1)$.

- let $\Gamma_0 := \phi$ and define $A_n = \Gamma_n - \Gamma_{n-1}$
then $A_k \cap A_l = \phi \quad \forall k \neq l$ and

$$\bigcup_1^\infty A_n = \bigcup_1^\infty \Gamma_n$$

$$\Rightarrow \mu\left(\bigcup_1^\infty \Gamma_n\right) = \sum_1^\infty \mu(A_n)$$

and

$$\underbrace{\sum_1^N \mu(A_n)}_{= \mu(\Gamma_N)} \uparrow \sum_1^\infty \mu(A_n)$$

$$\therefore \mu(\Gamma_N) \uparrow \sum_1^\infty \mu\left(\bigcup_1^\infty \Gamma_n\right).$$

- Let $A_n = \Gamma_1 - \Gamma_n \quad n \geq 2$
then $A_2 \subseteq A_3 \subseteq \ldots$ and by previous part,

$\mu(A_n) \uparrow \mu\left(\bigcup_2^\infty A_n\right)$ But $\bigcup_2^\infty A_n = \Gamma_1 - \bigcap_2^\infty \Gamma_n = \Gamma_1 - \bigcap_1^\infty \Gamma_k$

$$\Rightarrow \mu(\Gamma_1 - \Gamma_n) \uparrow \mu\left(\Gamma_1 - \bigcap_1^\infty \Gamma_k\right)$$

$$\mu(\Gamma_1) - \mu(\Gamma_n) \uparrow \mu(\Gamma_1) - \mu\left(\bigcap_1 \Gamma_k\right) \quad \text{since } \mu(\Gamma_1) < \infty,$$

$$\& \ \mu(\Gamma_n) \downarrow \mu\left(\bigcap_1^\infty \Gamma_k\right).$$

- Now let $\Gamma_k$ be arbitrary, rather than nested.
We want to show

$$\mu\left(\bigcup_1^\infty \Gamma_k\right) \leq \sum_1^\infty \mu(\Gamma_k).$$

Do this as follows:

$$\Gamma_0 := \emptyset \qquad A_{n+1} := \Gamma_{n+1} - \bigcup_1^n \Gamma_n \quad \text{for } n \geq 0$$

$$\Rightarrow \bigcup_1^n A_n = \bigcup_1^\infty \Gamma_n \quad \underline{\text{and}} \quad \Gamma_n = A_n \cup D_n \text{ where } D_n = \bigcup_{m=1}^{n-1} (\Gamma_n \cap \Gamma_m)$$

$$\rightarrow \mu\left(\bigcup_1^\infty \Gamma_n\right) = \mu\left(\bigcup_1^\infty A_n\right) = \sum_1^\infty \mu(A_n)$$

$$\leq \sum_1^\infty \mu(\Gamma_n)$$

this proves the desired inequality.

Furthermore $A_n \cap D_n = \emptyset \Rightarrow \mu(\Gamma_n) = \mu(A_n) + \mu(D_n)$

$$\Rightarrow \mu\left(\bigcup_1^\infty \Gamma_n\right) = \sum_1^\infty \mu(\Gamma_n) \Leftrightarrow \mu(D_n) = 0 \quad \forall n$$

since $\mu(D_n) \leq \sum_{k=1}^{n-1} \mu(\Gamma_n \cap \Gamma_k)$

we see that $\mu(\Gamma_n \cap \Gamma_k) = 0 \quad \forall n \neq k$ will imply what we need.

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We now have enough measure theory to define Lebesgue integrals.

To do this, we will need to be able to measure sets like $\{x \mid f(x) \in \Delta\}$ where $\Delta$ is a subset of the range of $f$.

Given measurable spaces $(E_1, \mathcal{B}_1)$ and $(E_2, \mathcal{B}_2)$ we say $\phi : E_1 \to E_2$ is a <u>measurable map</u> of $(E_1, \mathcal{B}_1)$ into $(E_2, \mathcal{B}_2)$ if

$$\{\phi \in \Gamma) := \{x \in E_1 \mid \phi(x) \in \Gamma\} = \phi^{-1}(\Gamma) \in \mathcal{B}_1 \quad \forall \Gamma \in \mathcal{B}_2.$$

Note the analogy with continuous maps! Instead of saying "$\phi^{-1}$(open) is open" we have "$\phi^{-1}$(measurable) is measurable"

<u>Lemma 3.2.1</u> Let $(E_1, \mathcal{B}_1)$ and $(E_2, \mathcal{B}_2)$ be measurable spaces and assume $\mathcal{B}_2 = \sigma(E_2; \mathcal{C})$ for some $\mathcal{C} \subseteq \mathcal{P}(E_2)$. If $\phi^{-1}(\Gamma) \in \mathcal{B}_1 \ \forall \Gamma \in \mathcal{C}$ then $\phi$ is measurable from $(E_1, \mathcal{B}_1)$ into $(E_2, \mathcal{B}_2)$. In particular, if $E_1$ and $E_2$ are topological spaces and $\mathcal{B}_1 = \mathcal{B}_{E_1}$ and $\mathcal{B}_2 = \mathcal{B}_{E_2}$ then every continuous map is measurable.

proof: exercise.

Note: this lemma is useful in that you just have to test the inverse images of the generating family $\mathcal{E}$! for example, if $(E_2, \mathcal{B}_2) = \mathbb{R}$ with $\mathcal{B}_{\mathbb{R}}$ then we just need to test $\phi^{-1}([\alpha, \infty))$ $\forall \alpha \in \mathbb{R}$.

Given $(E_1, \mathcal{B}_1)$ and $(E_2, \mathcal{B}_2)$ we define the measurable space $(E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)$ where

$$\mathcal{B}_1 \times \mathcal{B}_2 := \sigma\left(E_1 \times E_2 ; \{\Gamma_1 \times \Gamma_2 \mid \Gamma_1 \in \mathcal{B}_1, \Gamma_2 \in \mathcal{B}_2\}\right)$$

Similarly, if $\phi_1 : (E_0, \mathcal{B}_0) \to (E_1, \mathcal{B}_1)$ is measurable map and $\phi_2 : (E_0, \mathcal{B}_0) \to (E_2, \mathcal{B}_2)$ is measurable map then $\phi_1 \otimes \phi_2 : E_0 \to E_1 \times E_2$ defined by

$$\phi_1 \otimes \phi_2 (x) = (\phi_1(x), \phi_2(x)) \text{ is a measurable map}$$

Lemma 3.2.2: Assume $\phi_i$ is a measurable map on $(E_0, \mathcal{B}_0)$ into $(E_i, \mathcal{B}_i)$ for $i=1,2$.
Then $\phi_1 \otimes \phi_2$ is a measurable map on $(E_0, \mathcal{B}_0)$ into $(E_1 \otimes E_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$. Moreover, if $E_1$ and $E_2$ are second-countable topological spaces (the topology has a countable base) then

$$\mathcal{B}_{E_1} \otimes \mathcal{B}_{E_2} = \mathcal{B}_{E_1 \otimes E_2}$$

proof: see book.

One last thing before we define Lebesgue integrals. We want to allow functions to be more than real valued. i.e. $f: E \to [-\infty, \infty]$ should be integrable if $f$ isn't equal to $+\infty$ or $-\infty$ at too many points.

We will do this by defining

$$\overline{\mathbb{R}} = [-\infty, \infty] \text{ the \underline{extended} \underline{real} \underline{line}.}$$

The metric on $\overline{\mathbb{R}}$ will be defined by

$$\overline{\rho}(x,y) = \frac{2}{\pi}\left|\arctan(y) - \arctan(x)\right|$$

where $\arctan(\infty) := \frac{\pi}{2}$, $\arctan(-\infty) := -\pi/2$.
We have compactified $\mathbb{R}$ by adding $\pm\infty$. We see that $(\overline{\mathbb{R}}, \overline{\rho})$ is homeomorphic to $[-1, 1]$.

Since we have a metric on $\overline{\mathbb{R}}$, we give $\overline{\mathbb{R}}$ a topology and then define $\mathcal{B}_{\overline{\mathbb{R}}}$ (the Borel sets.)

We now define $\cdot$ and $+$ on $\overline{\mathbb{R}}$.

$$(\pm\infty)\cdot 0 := 0$$
$$0\cdot(\pm\infty) := 0$$
$$(\pm\infty)\cdot\alpha := \operatorname{sgn}(\alpha)\infty \quad \text{if } \alpha\in\overline{\mathbb{R}}-\{0\}$$
$$\alpha\cdot(\pm\infty) := \operatorname{sing}(\alpha)\cdot\infty \quad \text{" " "}$$

defined in this way, multiplication $\cdot : \overline{\mathbb{R}}^2 \to \overline{\mathbb{R}}$ is not continuous, but it is measurable.

defining addition is nastier since

$$\infty + (-\infty) = ??$$

So we'll define $+ : \hat{\mathbb{R}}^2 \to \overline{\mathbb{R}}$

where $\hat{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{(\infty, \infty), (-\infty, \infty)\}$.

$\widehat{\mathbb{R}^2}$ is an open subset of $\overline{\mathbb{R}}^2 \Rightarrow$

$$\mathcal{B}_{\widehat{\mathbb{R}^2}} = \mathcal{B}_{\overline{\mathbb{R}}^2}[\hat{\mathbb{R}}^2]$$

define $+ : \hat{\mathbb{R}}^2 \to \overline{\mathbb{R}}$ by

$$(\pm\infty) + \alpha = \alpha + (\pm\infty) = \pm\infty \quad \text{if } \alpha \neq \mp\infty$$

w/ this definition,

$+ : (\hat{\mathbb{R}}^2, \mathcal{B}_{\hat{\mathbb{R}}^2}) \to (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$ is continuous.

and hence measurable

Finally, $\vee : \overline{\mathbb{R}^2} \to \overline{\mathbb{R}}$

defined by
$$\begin{aligned} \alpha \vee \beta &= \max\{\alpha, \beta\} && \text{if } \alpha, \beta \in \mathbb{R} \\ &= \infty && \text{if } \alpha \text{ or } \beta = \infty \\ &= \alpha && \text{if } \beta = -\infty \\ &= -\infty && \text{if } \alpha = \beta = -\infty \end{aligned}$$

is continuous and hence measurable.

Similarly, $\wedge : \overline{\mathbb{R}}^2 \to \overline{\mathbb{R}}$ is continuous.

Now we can invoke our lemma 3.2.2. to say that

$f_1, f_2$ measurable functions from $(E, \mathcal{B})$ to $(\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$

then $f_1 \cdot f_2$ is measurable

$f_1 + f_2$ is measurable if range $(f_1 \times f_2) \subseteq \hat{\mathbb{R}}^2$

$f_1 \vee f_2$, $f_1 \wedge f_2$ are measurable.

It then follows that if $f$ is $\overline{\mathbb{R}}$-valued and measurable then

$$f^+ := f \vee 0, \quad f^- := f \wedge 0,$$

$$|f| := f^+ - f^-$$

are $\overline{\mathbb{R}}$-valued and measurable.

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Simple functions

$\phi : (E, \mathcal{B}) \to (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$ is simple if it takes only finitely many values. i.e. range $\{\phi\}$ is a finite set.

ex: $(E, \mathcal{B}) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and $\phi(x) = \begin{cases} 3 & x \in [1, 2) \\ \infty & x \in (4, 5] \\ 9 & \text{otherwise} \end{cases}$

the class of simple functions is closed under multiplication, $\vee$, $\wedge$, and (when defined) $+$.

the simplest simple function is the characteristic function: fix $\Gamma \subseteq E$.

$$1_\Gamma : (E, \mathcal{B}) \to (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$$

where $1_\Gamma(x) = \begin{cases} 1 & \text{if } x \in \Gamma \\ 0 & \text{if } x \notin \Gamma \end{cases}$

Note. $1_\Gamma$ is measurable $\iff \Gamma \in \mathcal{B}$.

Let $\phi$ be a non-negative, measurable, simple function. Then

<u>defn</u>: the Lebesgue integral of $\phi = \sum\limits_{\alpha \in \text{range}\,\phi} \alpha\, \mu(\phi = \alpha)$

recall $\mu(\phi = \alpha) = \mu(\{x \mid \phi(x) = \alpha\})$.

Notation:

$$\int_E f(x)\, \mu(dx), \quad \int_E f\, d\mu, \quad \int f\, d\mu$$

all represent this Lebesgue integral.

Further, if $\Gamma \in \mathcal{B}$ then

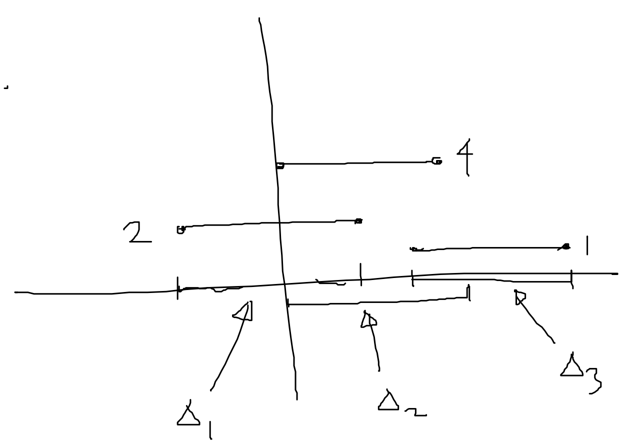
$$\int_\Gamma f(x)\, \mu(dx) \;:=\; \int_E f(x)\, 1_\Gamma(x)\, \mu(dx)$$

We now test the goodness of our definition via consistency results.

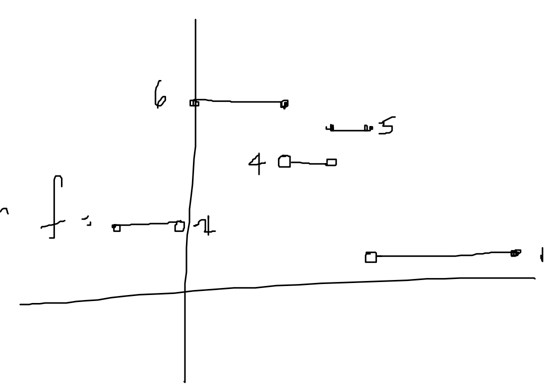
if $f = \sum_1^n \beta_\ell 1_{\Delta_\ell}$ where $\{\beta_\ell\}_1^n \subseteq [0,\infty]$, $\Delta_1, \ldots \Delta_n \in \mathcal{B}$

then we would like $\int f\, d\mu = \sum_1^n \beta_\ell\, \mu(\Delta_\ell)$.

i.e.




then $f =$




$$f = 2\,1_{\Delta_1} + 4\,1_{\Delta_2} + 1\cdot 1_{\Delta_3}$$

$$\int f\, d\mu = 2\cdot \mu(f=2) + 6\cdot\mu(f=6) + 5\,\mu(f=5) + 4\,\mu(f=4) + 1\,\mu(f=1)$$

does this equal
$2\,\mu(\Delta_1) + 4\,\mu(\Delta_2) + 1\,\mu(\Delta_3)$ ?

The proof is surprisingly gruesome given the simplicity of the hyper

proof:

We know $f$ achieves finitely many values, denote them $\{\alpha_1, \alpha_2 \dots \alpha_m\}$. Define $\Gamma_k = \{f = \alpha_k\}$ for $1 \le k \le m$. clearly $\Gamma_k \cap \Gamma_l = \emptyset$ if $k \ne l$.

$$\Rightarrow \sum_{l=1}^{n} \beta_l \mu(\Delta_l) = \sum_{l=1}^{n} \beta_l \sum_{k=1}^{m} \mu(\Delta_l \cap \Gamma_k) = \sum_{k=1}^{m} \sum_{l=1}^{n} \beta_l \mu(\Delta_l \cap \Gamma_k).$$

If we can show $\sum_{l=1}^{n} \beta_l \mu(\Delta_l \cap \Gamma_k) = \alpha_k \mu(\Gamma_k)$ then we're done! We do know $\alpha_k 1_{\Gamma_k} = \sum_{l=1}^{n} \beta_l 1_{\Delta_l \cap \Gamma_k}$ so we can work at the level of characteristic functions...

Since all the $\Gamma_k$ have the same role, we see that it suffices to show that if $\alpha \in [0,\infty]$ and $\Delta_1 \dots \Delta_n \in \mathcal{B}[\Gamma]$ then

$$\sum_{l=1}^{n} \beta_l 1_{\Delta_l} = \alpha 1_{\Gamma} \Rightarrow \sum_{l=1}^{n} \beta_l \mu(\Delta_l) = \alpha \mu(\Gamma).$$

where $\beta_1, \dots \beta_n \in [0,\infty]$. Note: if $\alpha = 0$ then $\beta_1 \dots \beta_n = 0$ and it's trivially true. So we'll consider $\alpha \in (0,\infty]$

Let $I = \{0,1\}^n$ and for $\vec{\eta} \in I$ define

$$\beta_{\vec{\eta}} = \sum_{1}^{n} \eta_l \beta_l \quad \text{and} \quad \Delta_{\vec{\eta}} = \bigcap_{l=1}^{n} \Delta_l^{(\eta_l)}$$

where $\Delta_l^{(1)} = \Delta_l$ and $\Delta_l^{(0)} = \Gamma - \Delta_l$. (Basically, we want to look at all possible intersections of $\Delta_l$ and $\Gamma - \Delta_l$ as $l$ ranges from 1 to $n$.)

then $\Delta_{\vec{\eta}} \cap \Delta_{\vec{\eta}'} = \phi$ if $\vec{\eta} \neq \vec{\eta}'$ by construction.

Also, $\Delta_l = \bigcup_{\vec{\eta} \mid \eta_l = 1} (\Delta_{\vec{\eta}})$ by construction.

$$\Rightarrow \sum_{\vec{\eta} \in I} \beta_{\vec{\eta}} 1_{\Delta_{\vec{\eta}}} = \sum_{\vec{\eta} \in I} \left( \sum_{l=1}^{n} \eta_l \beta_l \right) 1_{\Delta_{\vec{\eta}}}$$

$$= \sum_{l=1}^{n} \sum_{\vec{\eta} \in I} \eta_l \beta_l 1_{\Delta_{\vec{\eta}}}$$

$$= \sum_{l=1}^{n} \sum_{\vec{\eta} \in I, \eta_l = 1} \eta_l \beta_l 1_{\Delta_{\vec{\eta}}}$$

by assumption

$$= \sum_{l=1}^{n} \beta_l \sum_{\vec{\eta} \in I, \eta_l = 1} 1_{\Delta_{\vec{\eta}}} = \sum_{l=1}^{n} \beta_l 1_{\Delta_l} = \alpha 1_{\Gamma}$$

Since $\alpha 1_\Gamma = \sum_{\vec{\eta} \in I} \beta_{\vec{\eta}} 1_{\Delta_{\vec{\eta}}}$ we see that if $\Delta_{\vec{\eta}} \neq \phi$ then $\beta_{\vec{\eta}} = \alpha$

(because the $\Delta_{\vec{\eta}}$'s are disjoint.) And

$$\Gamma = \bigcup_{\vec{\eta} \in I'} \Delta_{\vec{\eta}} \quad \text{where } I' = \{ \vec{\eta} \in I \mid \Delta_{\vec{\eta}} \neq \phi \}$$

We can now move to measures since

$$\Gamma = \bigcup_{\vec{\eta} \in I'} \Delta_{\vec{\eta}}$$ where we have disjoint sets...

$$\sum_{l=1}^{n} \beta_l \mu(\Delta_l) = \sum_{l=1}^{n} \beta_l \sum_{\{\vec{\eta} \mid \eta_l = 1\}} \mu(\Delta_{\vec{\eta}}) = \sum_{\vec{\eta} \in I'} \beta_{\vec{\eta}} \mu(\Delta_{\vec{\eta}})$$

$$= \alpha \sum_{\vec{\eta} \in I'} \mu(\Delta_{\vec{\eta}}) = \alpha \mu(\Gamma)$$ as desired.

That lemma is actually useful. It yields

__Lemma 3.2.4__: Let $f$ and $g$ be non-negative simple measurable functions on $(E, \mathcal{B}, \mu)$. Then for any $\alpha, \beta \in [0, \infty]$ $\alpha f + \beta g$ is a non-negative simple measurable function and

$$\int (\alpha f + \beta g)\, d\mu = \alpha \int f\, d\mu + \beta \int g\, d\mu$$

In particular, if $f \leq g$ then $\int f\, d\mu \leq \int g\, d\mu$.
Furthermore if $f \leq g$ __and__ $\int f\, d\mu < \infty$ then

$$\int (g - f)\, d\mu = \int g\, d\mu - \int f\, d\mu.$$

__Note!__ I've dropped the $\overline{\mathbb{R}}$-valued from "non-negative simple measurable functions" since it's implicit.

proof:

1) $f$ non negative $\mathcal{B}$ simple $\mathcal{B}$ measurable

$$\Rightarrow f = \sum_1^m \alpha_k 1_{\Delta_k} \quad \text{for} \quad \{\alpha_1, \ldots \alpha_m\} \subset [0,\infty]$$
$$\{\Delta_1 \ldots \Delta_m\} \subset \mathcal{B}.$$

similarly,

$$g = \sum_{k=m+1}^{n+m} \beta_k 1_{\Delta_k} \quad \text{for} \quad \{\beta_{m+1} \ldots \beta_{m+n}\} \subset [0,\infty]$$
$$\{\Delta_{m+1}, \ldots \Delta_{m+n}\} \subset \mathcal{B}.$$

$$\rightarrow \alpha f + \beta g = \sum_{k=1}^{n+m} \gamma_k 1_{\Delta_k} \quad \text{where} \quad \gamma_k = \alpha \alpha_k \quad \text{if } k \in \{1 \ldots m\}$$
$$\gamma_k = \beta \beta_k \quad \text{if } k \in \{m+1 \ldots n+m\}$$

By lemma 3.2.3,

$$\int \alpha f + \beta g \, d\mu = \sum_{k=1}^{n+m} \gamma_k \mu(\Delta_k)$$

$$= \alpha \sum_{k=1}^{n} \alpha_k \mu(\Delta_k) + \beta \sum_{k=m+1}^{m+n} \beta_k \mu(\Delta_k)$$

$$= \alpha \int f \, d\mu + \beta \int g \, d\mu$$

again by lemma 3.2.3 !

2) Assume $f \leq g$. $f = \sum_{k=1}^{m} \alpha_k 1_{\Delta_k}$ $g = \sum_{k=m+1}^{m+n} \beta_k 1_{\Delta_k}$ WLOG $\alpha_k \neq 0$, $\beta_k \neq 0$.

$\Rightarrow$ if $x \in \Delta_k$ for some $k = 1 \ldots m$

then $f(x) = \alpha_k > 0$ $\rightarrow$ $x \in \Delta_l$ some $l = m+1, \ldots m+n$ where $\alpha_k \leq \beta_l$

So $f \leq g \Rightarrow \Delta_k \subseteq \bigcup_{m+1}^{m+n} \Delta_l$

and if $x \in \Delta_k \cap \Delta_l$ then $f(x) = \alpha_k \leq g(x) = \alpha_l$.

$$\Delta_k = \bigcup_{l=m+1}^{m+n} \underbrace{(\Delta_k \cap \Delta_l)}$$

mutually disjoint sets

$$\Rightarrow \int f d\mu = \sum_{k=1}^{m} \alpha_k \mu(\Delta_k) = \sum_{k=1}^{m} \alpha_k \sum_{l=m+1}^{m+n} \mu(\Delta_k \cap \Delta_l)$$

lemma 3.2.3

$$= \sum_{k=1}^{m} \sum_{l=m+1}^{m+n} \alpha_k \mu(\Delta_k \cap \Delta_l)$$

$$\leq \sum_{l=m+1}^{m+n} \sum_{k=1}^{m} \alpha_l \mu(\Delta_k \cap \Delta_l)$$

$$= \sum_{l=m+1}^{m+n} \alpha_l \sum_{k=1}^{m} \mu(\Delta_k \cap \Delta_l)$$

$$\leq \sum_{l=m+1}^{m+n} \alpha_l \mu(\Delta_l)$$

since $\bigcup_{k=1}^{m} (\Delta_k \cap \Delta_l) \subseteq \Delta_l$

$$= \int g d\mu$$

lemma 3.2.3.

•) if $\int g d\mu < \infty$ and $f \leq g \Rightarrow \int g - f d\mu = \int g d\mu - \int f d\mu$

exercise.

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