Distributions, Test Function Spaces.

Very generally speaking, if

(X, τ) is a topological vector space where members of X are functions, then

$f \in X^*$ is "a distribution on X".

(Recall X^* = continuous real-valued linear fxnls on X

or = continuous complex-valued " " " .)

What we're going to do is consider different choices of (X, τ). Here are three very famous examples:

1) $X = C^\infty(\mathbb{R}) = \{ \phi : \mathbb{R} \to \mathbb{R} \mid \phi^{(k)}$ exists for each $k \geq 1 \}$

τ = ?? (will discuss later)

$\phi(x) = 1$

$\phi(x) = \sin(x)$ are all in $C^\infty(\mathbb{R})$

$\phi(x) = e^{x^2}$

2) $X = \mathcal{S}(\mathbb{R}) = \{ \phi : \mathbb{R} \to \mathbb{R} \mid |x|^k \phi^{(m)}(x) \to 0$ as $|x| \to \infty$ for each $k \geq 1$, $m \geq 0 \}$

τ = ?? (will discuss later)

So $\phi \in \mathcal{S}(\mathbb{R})$ if ϕ and all of its derivatives decay as $|x| \to \infty$. And they decay faster than any polynomial

$$\phi(x) = e^{-x^2} \in \mathcal{S}(\mathbb{R}^n)$$

3) $X = C_0^\infty(\mathbb{R}) = \left\{ \phi : \mathbb{R} \to \mathbb{R} \;\middle|\; \phi^{(k)} \text{ exists } \forall k \geq 1 \text{ and } \phi \text{ has compact support.} \right\}$

$\tau = ??$ (will discuss later)

Note: $\text{Support}(\phi) := \overline{\{x \in \mathbb{R} \mid \phi(x) \neq 0\}}$

What's an example of $\phi \in C_0^\infty$?? (will discuss later.)

In any case, it's clear that

$$C_0^\infty \subseteq \mathcal{S}(\mathbb{R}) \subseteq C^\infty(\mathbb{R})$$

if the topologies on $C_0^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, and $C^\infty(\mathbb{R})$ are compatible (in some sense... will discuss later) then

$$(C^\infty(\mathbb{R}))^* \subseteq (\mathcal{S}(\mathbb{R}))^* \subseteq (C_0^\infty(\mathbb{R}))^*$$

So here I've defined three topological vector spaces whose members are functions (A.K.A. three "test function spaces") and as a result, I'll have three different types of distributions right there.

Q: What type of X should I choose?
A: It depends what you're trying to do...

Let's consider some specific distributions and we'll ask, in each case, what sort of constraints we need on our test functions to make any sense of the distribution.

ex 1: the delta function centered at $x=0$

$$\delta_0(\phi) = \phi(0)$$

we want $\delta_0 : X \to \mathbb{R}$ ✓

δ_0 linear ✓

δ_0 continuous (what is τ?)

ex: the improper integral

$$\phi \longrightarrow \int_{-\infty}^{\infty} \phi(x)\,dx$$

ex 3: the Fourier transform

$$\phi \to \hat{\phi}(\xi) := \int_{-\infty}^{\infty} \phi(x) e^{-ix\xi} dx$$ ← note: $\hat{\phi}(\xi) \in \mathbb{C}$. but no big deal here.

ex 4: the Cauchy Principal Value

$$\phi \to \lim_{\varepsilon \downarrow 0} \int_{|x|>\varepsilon} \frac{\phi(x)}{x} dx$$

Now what's the big deal? These are "just" integrals in the last three examples. Well if we define integration in terms of areas

$$\int \phi(x)\,dx = \int \phi_+(x)\,dx + \int \phi_-(x)\,dx$$

= area above graph
− area below graph

then we're hosed. Why? We want to be able to take advantage of cancellation effects!

recall $\sum_1^\infty \frac{(-1)^n}{n}$ this series converges

even though $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$

and $\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$

both diverge.

We can't define

$$\phi \to \hat{\phi}(\xi) = \int \phi(x) e^{-ix\xi} dx$$

for $\phi \equiv$ constant function unless we can take advantage of the cancellations that multiplying by $e^{-ix\xi}$ give us.

Okay, let's look at these examples more closely.

First the δ function. We've already seen that we can define δ_0 if $X =$ space of continuous, bounded functions, with L^∞ norm.

So our intuition is that for us to be able to make sense of $\phi \to \phi(0)$ we'll just need that ϕ is bounded on $\mathbb{R}$.

Let's revive interest in the humble δ-function by looking at a specific application.

<u>The Heat Equation</u>

$u(x,t)$ is a solution of the heat equation on the line if

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$$

for all $x \in \mathbb{R}$ and all $t \in \mathbb{R}$.

On a more practical level, we're interested in solutions of the heat equation as an initial value problem.

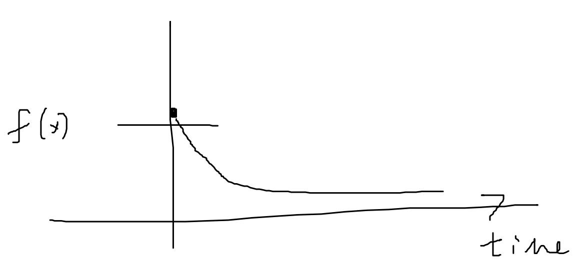
That is, if at time $t=0$ the distribution of temperature is $f(x)$, what will the temperature distribution be at $t=1$?

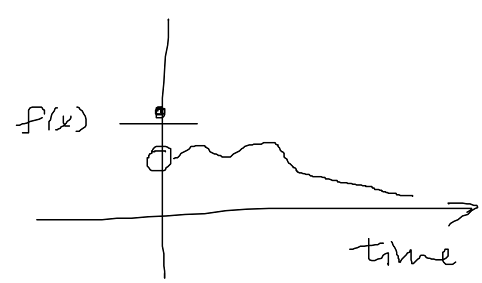
Heat equation as initial value problem:

$$(1) \quad \frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \qquad \forall x \in \mathbb{R},\ \forall t > 0$$

$$(2) \quad \lim_{t \downarrow 0} u(x,t) = f(x) \qquad \forall x \in \mathbb{R}$$

(2) is the initial data. That looks really wierd. Why don't we just say "$u(x,0) = f(x)$"? Well, 2) is stronger! It tells us that there's continuity in time at $t=0$!





Condition (2) says this.

Also, $u(x,t)$ is only defined for $t > 0$.

Okay... how do we get a solution then? we build it using an "exact" solution called a point-source solution

$$E(x,t) := \frac{1}{\sqrt{4\pi t}}\, e^{\frac{-x^2}{4t}}$$

Fact: $\quad \dfrac{\partial E}{\partial t} = \dfrac{\partial^2 E}{\partial x^2} \quad \forall x \in \mathbb{R},\ \forall t > 0$

Fact: $\quad \displaystyle\int_{-\infty}^{\infty} E(x,t)\,dx = 1 \quad \forall t > 0$

Fact: $\quad \displaystyle\lim_{t \downarrow 0} E(x,t) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$

Hmmm... this looks familiar... if we define

$$\phi \longrightarrow F_t(\phi) = \int_{-\infty}^{\infty} E(x,t)\,\phi(x)\,dx$$

then $F_t \xrightarrow[w*]{} \delta_0$ probably. (we did something like this with

 piecewise linear functions...)

Return to the initial value problem

$$\circledast \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & \forall x,\ \forall t > 0 \\ \displaystyle\lim_{t \downarrow 0} u(x,t) = f(x) & \forall x \end{cases}$$

claim.

$$u(x,t) := \int_{-\infty}^{\infty} E(y,t) f(x-y)\, dy$$

solves the initial value problem.

Q: what constraints do we need to put on f in order to know that

$$(x,t) \longrightarrow u(x,t)$$

makes sense for all $x \in \mathbb{R}$, $t > 0$

and to know that $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

and to know that $\lim_{t \downarrow 0} u(x,t) = f(x)$?

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} E(y,t) f(x-y)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{\partial E}{\partial t}(y,t) f(x-y)\, dy$$

careful! differentiated under $\int$ sign!

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \int E(y,t) f(x-y)\, dy = \int E(y,t) \frac{\partial}{\partial x} f(x-y)\, dx$$

$$= - \int E(y,t) \frac{\partial}{\partial y} f(x-y)\, dy$$

$$= - E(y,t) f(x-y) \Big|_{y=-\infty}^{y=+\infty} + \int \frac{\partial E}{\partial y}(y,t) f(x-y)\, dy$$

if we know $E(y,t) f(x-y) \to 0$ as $|y| \to \infty$
then the boundary terms vanish. One way
to know this is if $f(x) \to 0$ as $|x| \to \infty$ and
if f' and $f'' \to 0$ as $|x| \to \infty$

($f \to 0$ ensures that $E(y,t) f(x-y)$ is integrable for sure.)

$$\Rightarrow \frac{\partial u}{\partial x} = \int_{-\infty}^{\infty} \frac{\partial E}{\partial y}(y,t) f(x-y)\, dy$$

Similarly, $$\frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} \frac{\partial^2 E}{\partial y^2}(y,t) f(x-y)\, dy$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} \left[\frac{\partial E}{\partial t} - \frac{\partial^2 E}{\partial y^2} \right] f(x-y)\, dy$$

$$= 0 \quad \text{since} \quad \frac{\partial E}{\partial t} = \frac{\partial^2 E}{\partial y^2}$$

$\Rightarrow$ we know $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for all $x \in \mathbb{R}$, $\forall t > 0$.

Prop. Assume ϕ is continuous on $\mathbb{R}$
and $\sup_{x \in \mathbb{R}} |\phi(x)| < \infty$.
then $$\lim_{t \downarrow 0} \int_{-\infty}^{\infty} E(y,t) \phi(y)\, dy = \phi(0).$$

corollary: if f is cts and bounded on $\mathbb{R}$

then $\lim_{t \downarrow 0} u(x,t) = f(x)$.

proof of proposition:

$$\int_{-\infty}^{\infty} E(y,t)\,\phi(y)\,dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} \phi(\sqrt{4t}\,u)\,du$$

if $y = \sqrt{4t}\,u$

want

$$\lim_{t \downarrow 0} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} \phi(\sqrt{4t}\,u)\,du - \phi(0) = 0$$

recall, $\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$.

$\Rightarrow$ want

$$\lim_{t \downarrow 0} \frac{1}{\sqrt{\pi}} \int e^{-u^2} \phi(\sqrt{4t}\,u)\,du - \frac{1}{\sqrt{\pi}} \int e^{-u^2} \phi(0)\,du = 0$$

$\Rightarrow$ want

$$\lim_{t \downarrow 0} \frac{1}{\sqrt{\pi}} \int e^{-u^2} \left[\phi(\sqrt{4t}\,u) - \phi(0)\right] du = 0$$

we have

$$\int_{-\infty}^{\infty} e^{-u^2}[\phi(\sqrt{4t}\,u)-\phi(0)]\,du$$

since the integral converges, we know its tails

$\int_{-\infty}^{-M}$ and $\int_{M}^{\infty}$ can be made small.

And if $|t| <$ small ~~#~~ then $\phi(\sqrt{4t}\,u)-\phi(0)$
will be small for $u \in [-M, M]$ and
this will allow us to control $\int_{-M}^{M}$ and make it small.

Given $\varepsilon > 0$ we want to find $\delta \ni 0 < t < \delta$

then $\left| \int_{-\infty}^{\infty} e^{-u^2}[\phi(\sqrt{4t}\,u)-\phi(0)]\,du \right| < \varepsilon$

Step 1. Let $K = \sup_{x \in \mathbb{R}} |\phi(x)|$

then $|\phi(\sqrt{4t}\,u)-\phi(0)| \leq 2K$. Choose

M sufficiently large so that

$$\left| \int_{M}^{\infty} e^{-u^2}[\phi(\sqrt{4t}\,u)-\phi(0)]\,du \right| \leq 2K\int_{M}^{\infty} e^{-u^2}\,du < \varepsilon/3$$

similarly, $\left| \int_{-\infty}^{-M} e^{-u^2}[\phi(\sqrt{4t}\,u-\phi(0)]\,du \right| < \varepsilon/3$

Now that M is fixed, we choose t small to make $\left| \int_{-M}^{M} e^{-u^2} [\phi(\sqrt{4t}\,u) - \phi(0)]\, du \right| < \varepsilon/3$

we know ϕ is cts. specifically it's cts at 0

$\Rightarrow \exists\, \delta$ so that $|y| < \delta \Rightarrow |\phi(y) - \phi(0)| < \frac{\varepsilon}{3} \frac{1}{\sqrt{\pi}}$

$\Rightarrow$ If t is such that

$$|\sqrt{4t}\,u| < \delta \quad \text{for all } u \in [-M, M]$$

then we have control of

$$|\phi(\sqrt{4t}\,u) - \phi(0)| \quad \text{for } u \in [-M, M].$$

i.e. $\sqrt{4t} < \delta/M \Rightarrow 4t < \delta^2/M^2 \Rightarrow t < \frac{\delta^2}{4M^2}$

Assume $0 < t < \frac{\delta^2}{4M^2}$ then

$$\left| \int_{-M}^{M} e^{-u^2} [\quad]\, du \right| \le \int_{-M}^{M} e^{-u^2} |\phi(\sqrt{4t}\,u) - \phi(0)|\, du$$

$$\le \int_{-M}^{M} e^{-u^2} \cdot \frac{\varepsilon}{3} \frac{1}{\sqrt{\pi}}\, du < \frac{\varepsilon}{3} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\, du$$

$$= \frac{\varepsilon}{3}.$$

This proves $\lim_{t \downarrow 0} \int_{-\infty}^{\infty} E(y,t)\,\phi(y)\,dy = \phi(0)$, as desired. //

What we've (nearly) shown is that

$$E(\cdot,t) \xrightarrow[w*]{} \delta_0$$

(we haven't formally defined the space of test functions X and its topology yet.) Basically we have $E(\cdot,t)$ "converges to δ_0 in the sense of distributions" is another way of saying "converges weak *".

Another example!

the Cauchy Principal value

$$\phi \to \lim_{\varepsilon \downarrow 0} \left(\int_{-\infty}^{-\varepsilon} \frac{\phi(x)}{x}\,dx + \int_{\varepsilon}^{\infty} \frac{\phi(x)}{x}\,dx \right) =: PV \tfrac{1}{x}(\phi)$$

Condition 1 on ϕ: ϕ decays fast enough at $\pm\infty$ so that $\int_1^\infty \frac{\phi(x)}{x}\,dx$ and $\int_{-\infty}^{-1} \frac{\phi(x)}{x}\,dx$ are finite

$\Rightarrow$ only have to worry about $x=0$

We will study $PV\frac{1}{x}(\phi)$ in two steps

1) show $\left[\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right]$ is bounded as $\varepsilon \downarrow 0$

Step 2 show the limit exists by guessing the limit and then proving that as $\varepsilon \downarrow 0$ $\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}$ converges to our guess.

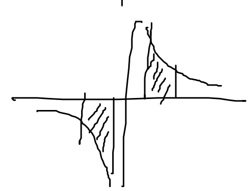
$$\int_{|x|>\varepsilon} \frac{\phi(x)}{x}\,dx = \int_{-\infty}^{-1} \frac{\phi(x)}{x}\,dx + \int_{-1}^{-\varepsilon} \frac{\phi(x)-\phi(0)}{x}\,dx + \int_{-1}^{-\varepsilon} \frac{\phi(0)}{x}\,dx$$

$$+ \int_{\varepsilon}^{1} \frac{\phi(x)-\phi(0)}{x}\,dx + \int_{\varepsilon}^{1} \frac{\phi(0)}{x}\,dx + \int_{1}^{\varepsilon} \frac{\phi(x)}{x}\,dx$$

Note: $\displaystyle\int_{-1}^{-\varepsilon} \frac{\phi(0)}{x}\,dx + \int_{\varepsilon}^{1} \frac{\phi(0)}{x}\,dx = \phi(0)\left\{ \ln|x| \Big|_{1}^{-\varepsilon} + \ln|x| \Big|_{\varepsilon}^{1} \right\}$

$$= 0$$

oooh! clever! we used the cancellation of $\frac{1}{x}$



assume $K := \sup_{x \in \mathbb{R}} |x\phi(x)| < \infty$

this will control the $\int_{-\infty}^{-1}$ and $\int_{1}^{\infty}$ integrals

as follows.

$$\left|\int_{-\infty}^{-1} \frac{\phi(x)}{x}dx\right| \le \int_{-\infty}^{-1} \left|\frac{\phi(x)}{x}\right|dx = \int_{-\infty}^{-1} \left|\frac{x\phi(x)}{x^2}\right|dx$$

$$\le K\int_{-\infty}^{-1} \frac{1}{x^2}dx < \infty.$$

similarly, we control $\left|\int_{1}^{\infty} \frac{\phi(x)}{x}dx\right|$

we now control

$$\int_{-1}^{-\varepsilon} \frac{\phi(x)-\phi(0)}{x}dx + \int_{\varepsilon}^{1} \frac{\phi(x)-\phi(0)}{x}dx$$

$$= \int_{\varepsilon}^{1} \frac{\tilde\phi(x)-\tilde\phi(-x)}{x}dx \quad \text{where } \tilde\phi(x) = \phi(x)-\phi(0)$$

By the mean value theorem (need ϕ to have a deriv!)

$$\tilde\phi(x)-\tilde\phi(-x) = 2x\,\tilde\phi'(c_x) \quad \text{some} \quad c_x \in (-x,x)$$

$$\Rightarrow \left|\int_{\varepsilon}^{1} \frac{\tilde\phi(x)-\tilde\phi(-x)}{x}dx\right| \le \int_{\varepsilon}^{1} \left|\frac{2x\,\tilde\phi'(c_x)}{x}\right|dx = \int_{\varepsilon}^{1} 2|\phi'(c_x)|dx$$

assume ϕ' is cts and $K_1 := \sup_{y\in\mathbb{R}} |\phi'(y)| < \infty$

then $\left| \int_\varepsilon^1 \frac{\tilde\phi(x)-\tilde\phi(-x)}{x}\,dx \right| \le \int_\varepsilon^1 2K_1\,dx < 2K_1 < \infty.$

this finishes part 1:

$$\left| \int_{|x|>\varepsilon} \frac{\phi(x)}{x}\,dx \right| \le \left| \int_{-\infty}^{-1} \frac{\phi(x)}{x}\,dx \right| + \left| \int_1^\infty \frac{\phi(x)}{x}\,dx \right| + \left| \int_\varepsilon^1 \frac{\tilde\phi(x)-\tilde\phi(-x)}{x}\,dx \right| < \infty.$$

So we've <u>bounded</u> $\int_{|x|>\varepsilon} \frac{\phi(x)}{x}\,dx$, as a fxn of ε

Now we want to show it has a limit as $\varepsilon \downarrow 0$. We expect the limit is

$$\int_{-\infty}^{-1} \frac{\phi(x)}{x}\,dx + \int_1^\infty \frac{\phi(x)}{x}\,dx + \lim_{\varepsilon\downarrow 0} \left[\int_{-1}^{-\varepsilon} \frac{\tilde\phi(x)}{x}\,dx + \int_\varepsilon^1 \frac{\tilde\phi(x)}{x}\,dx \right]$$

Now recall $\tilde\phi(x) = \phi(x) - \phi(0)$

by taylor's expansions, $\tilde\phi(x)$ has an expansion $\phi'(0)x + \text{H.O.T.}$ $\Rightarrow \frac{\tilde\phi(x)}{x} = \phi'(0) + \text{H.O.T.}$

$\frac{\tilde\phi(x)}{x}$ is a nice continuous function!

So since $\frac{\tilde{\phi}(x)}{x}$ is continuous,

$$\lim_{\varepsilon \downarrow 0} \left[\int_{-1}^{-\varepsilon} \frac{\tilde{\phi}(x)}{x} dx + \int_{\varepsilon}^{1} \frac{\tilde{\phi}(x)}{x} dx \right] \text{ should } = \int_{-1}^{1} \frac{\tilde{\phi}(x)}{x} dx$$

$$= \int_{-1}^{1} \frac{\phi(x) - \phi(0)}{x} dx.$$

Let's prove this! We want to prove

$$\lim_{\varepsilon \downarrow 0} \left[\int_{-1}^{-\varepsilon} \frac{\tilde{\phi}(x)}{x} dx + \int_{\varepsilon}^{1} \frac{\tilde{\phi}(x)}{x} dx - \int_{-1}^{1} \frac{\tilde{\phi}(x)}{x} dx \right] = 0$$

i.e.

$$\lim_{\varepsilon \downarrow 0} \left[\int_{-\varepsilon}^{\varepsilon} \frac{\tilde{\phi}(x)}{x} dx \right] = 0$$

Since $x \to \frac{\tilde{\phi}(x)}{x}$ is a cts fxn of x, $\exists$ M s.t.

that $\left| \frac{\tilde{\phi}(x)}{x} \right| \leq M < \infty$ if $|x| < 1$.

$$\Rightarrow \left| \int_{-\varepsilon}^{\varepsilon} \frac{\tilde{\phi}(x)}{x} dx \right| \leq \int_{-\varepsilon}^{\varepsilon} \left| \frac{\tilde{\phi}(x)}{x} \right| dx \leq M \int_{-\varepsilon}^{\varepsilon} dx = 2\varepsilon M.$$

This shows that $\lim_{\varepsilon \downarrow 0} [\ \ ] = 0$, as desired.

This finishes the proof that

if ϕ satisfies "Assumptions 1 ... 4"

then

$$\phi \to PV\ \frac{1}{x}(\phi) := \lim_{\varepsilon \downarrow 0} \int_{|x|>\varepsilon} \frac{\phi(x)}{x}\,dx$$

is defined. (It's certainly linear too! The next question is ... is it continuous?)