Inverse + Adjoint operators.

given $A : E \to E_1$

if $R_A \subseteq E_1$ is the range of A and

if A is 1:1 then A^{-1} exists.

Theorem: if A is a linear operator then A^{-1} is a linear operator.

proof: see K+F.

Theorem: Let A be an invertible bounded linear operator $A : E \to E_1$ where E & E_1 are Banach spaces. Then A^{-1} is a bounded operator

proof 1: we just did it

proof 2: see K+F and see how their proof relates to the open mapping theorem

Theorem: Let A_0 be an invertible bounded linear operator $A_0 : E \to E_1$ where E and E_1 are Banach spaces. Assume ΔA is another bounded linear operator such that

$$\| \Delta A \|_{\mathcal{L}(E,E_1)} < \frac{1}{\| A_0^{-1} \|_{\mathcal{L}(E_1,E)}}$$

then $A_0 + \Delta A$ is a bounded operator $E \to E_1$ and has a bounded inverse.

Note: this theorem is saying that if you have an invertible operator and you don't perturb it too much then it will remain invertible.

e.g. $A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$

$A_0^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/10 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ $\qquad \|A_0^{-1}\| = 2$

now imagine $\Delta A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ $\qquad \|\Delta A\| = \max\{|\lambda_i|\}$

if $\|\Delta A\| < \dfrac{1}{\|A_0^{-1}\|} = \dfrac{1}{2}$ then $\max\{|\lambda_i|\} < \frac{1}{2}$

$\rightarrow A + \Delta A = \begin{pmatrix} 1+\lambda_1 & 0 & 0 \\ 0 & 10+\lambda_2 & 0 \\ 0 & 0 & -1/2 + \lambda_3 \end{pmatrix}$

As long as $|\lambda_3| < 1/2$ we're safe. and this is certainly true.

proof of theorem:

fix $y \in E_1$. Define

$$B : E \to E \text{ by}$$

$$Bx = A_0^{-1} y - A_0^{-1} (\Delta A) x$$

If we can find! $\tilde{x}$ so that $B\tilde{x} = \tilde{x}$ then

$$\tilde{x} = A_0^{-1} y - A_0^{-1} \Delta A \tilde{x} = A_0^{-1} (y - \Delta A \tilde{x})$$

$$\to A_0 \tilde{x} = y - \Delta A \tilde{x} \Rightarrow (A_0 + \Delta A) \tilde{x} = y \text{ and}$$

we've inverted $A_0 + \Delta A$

also since $A_0 + \Delta A$ is a bounded linear operator that is 1:1 and onto and then by previous theorem $(A_0 + \Delta A)^{-1}$ will be bounded operator. So all we need to do is find that unique $\tilde{x}$

Do this w/ contraction mapping theorem.

$$\| Bx_0 - Bx_1 \|_E = \| A_0^{-1} y - A_0^{-1} \Delta A x_0 - A_0^{-1} y + A_0^{-1} \Delta A x_1 \|_E$$

$$= \| A_0^{-1} \Delta A x_1 - A_0^{-1} \Delta A x_0 \|_E$$

$$\leq \| A_0^{-1} \| \, \| \Delta A x_1 - \Delta A x_0 \|_{E_1}$$

$$\leq \| A_0^{-1} \| \, \| \Delta A \| \, \| x_1 - x_0 \|_E \leq \alpha \| x_1 - x_0 \|_E$$

where $\alpha < 1$. $\Rightarrow \exists!$ fixed point and done. //

Q: Can we ever construct an inverse?

thm:

assume E is a Banach space and $A: E \to E$ a bounded linear operator such that $\|A\| < 1$.

then $(I - A)^{-1}$ exists, is bounded, and can be represented as

$$(I - A)^{-1} = \sum_{0}^{\infty} A^k \quad (*)$$

WHOAH! Geometric series for operators!

$$\frac{1}{1-x} = \sum_{0}^{\infty} x^k \quad \text{converges for } x \in \mathbb{R} \text{ if } |x| < 1.$$

proof: since $I^{-1} = I$ and $\frac{1}{\|I^{-1}\|} = 1$

we have $\|A\| < \frac{1}{\|I^{-1}\|} \Rightarrow$

the existence and boundedness of $I - A$ is by the previous theorem. We just need the representation $(*)$

Note: since $\|AB\| \leq \|A\| \|B\|$ for any $A, B \in \mathcal{L}(E, E)$ we have

$$\|A^k\| \leq \|A\|^k$$

$$\Rightarrow \sum_0^\infty \|A^k\| \leq \sum_0^\infty \|A\|^k < \infty \quad \text{since } \|A\|<1.$$

since $\sum_0^\infty \|A^k\| < \infty$ and $\mathcal{L}(E,E)$ is complete,

this implies that $\sum_0^n A^k$ converges

to an element of $\mathcal{L}(E,E)$ as $n\to\infty$. i.e.

$\sum_0^\infty A^k$ is a bounded linear operator from E to E

Now we just want to check that

$\sum_0^\infty A^k$ is the inverse of $I-A$.

we do this just like w/ geometric series.

let $B_n = \sum_0^n A^k$ we want to show

that as $n\to\infty$, $B_n \to (I-A)^{-1}$.

$$(I-A)B_n = (I-A)\sum_0^n A^k = I - A^{n+1}$$

$$\Rightarrow \|(I-A)B_n - I\| = \|A^{n+1}\| \leq \|A\|^{n+1}$$ and RHS $\to 0$ as $n\to\infty$

$\Rightarrow (I-A)B_n \to I$ as $n\to\infty$

$$\Rightarrow \sum_0^\infty A^k = (I-A)^{-1}. \quad //$$

Adjoint operators.

Recall from linear algebra:

A an $n \times m$ matrix

$A : \mathbb{R}^m \to \mathbb{R}^n$

then we define the adjoint of A, $A^* : \mathbb{R}^n \to \mathbb{R}^m$ by using the inner product:

$$\langle Ax, y \rangle_{\mathbb{R}^n} = \langle x, A^* y \rangle_{\mathbb{R}^m} \qquad \begin{array}{l} \text{for all } y \in \mathbb{R}^n \\ \text{for all } x \in \mathbb{R}^m \end{array}$$

if A is real-valued then $A^* = A^t$ $\quad (A^*)_{ij} = A_{ji}$

Q: How do we go to ∞ dimensions?

A: It's clear what to do if

$A : E \to E_1$

where E and E_1 have inner products. But what if they don't have inner products?

let E and E_1 be topological vector spaces and $A : E \to E_1$ a continuous linear operator.

fix $\phi \in E_1^*$

then $\phi(Ax) \in \mathbb{R}$

and $x \to \phi(Ax)$ is a continuous linear real-valued functional on E_1

i.e. $x \to \phi(Ax)$ is in E^*

$\Rightarrow \exists \psi \in E^*$ so that $x \to \psi(x)$ is the same as $x \to \phi(Ax)$

$$(\phi, Ax) = (\psi, x) \quad \forall x$$

we call ψ $A^*\phi$

i.e. $(\phi, (Ax) = (A^*\phi, x) \quad \forall x$

i.e. $\phi(Ax) = A^*\phi\,(x) \quad \forall x$

($A^*\phi$ is the pull-back of ϕ.)

$A^*: E_1^* \to E^*$

claim: A^* is linear.

$$A^*(\phi + \tilde{\phi}) \overset{?}{=} A^*\phi + A^*\tilde{\phi}$$

know $\left(A^*(\phi + \tilde{\phi})\right)(x) = (\phi + \tilde{\phi})(Ax) \quad \forall x \in E$

$$= \phi(Ax) + \tilde{\phi}(Ax)$$

$$= A^*\phi(x) + A^*\tilde{\phi}(x)$$

true $\forall x \in E \Rightarrow A^*(\phi + \tilde{\phi}) = A^*\phi + A^*\tilde{\phi}$.

check $A^*(\alpha\phi) = \alpha A^*\phi$ similarly
$= (\alpha A)^*\phi$

and $(A+B)^* = A^* + B^*$

theorem: Let $A \in \mathcal{L}(E, E_1)$ where E and E_1 are Banach spaces and let A^* be ~~the~~ the adjoint of A, $A^*: E_1^* \to E^*$ then A^* is bounded and

$$\|A^*\|_{\mathcal{L}(E_1^*, E^*)} = \|A\|_{\mathcal{L}(E, E_1)}$$

<u>proof</u>: fix $x \in E$

$$|A^* \phi(x)| = |\phi(Ax)| \leq \|\phi\|_{E_1^*} \|Ax\|_{E_1}$$
$$\leq \|\phi\|_{E_1^*} \|A\|_{\mathcal{L}(E,E_1)} \|x\|_E$$

this is true $\forall x \in E$

$$\Rightarrow \|A^* \phi\|_{E^*} \leq \|A\|_{\mathcal{L}(E,E_1)} \|\phi\|_{E_1^*}$$

$$\Rightarrow \|A^*\|_{\mathcal{L}(E_1^*, E^*)} \leq \|A\|_{\mathcal{L}(E,E_1)}$$

Now in the other direction. take $x_0 \in E$ so that $Ax_0 \neq \vec{0}$.

define $y_0 = \dfrac{Ax_0}{\|Ax_0\|_{E_1}} \in E_1$.

use y_0 to define $g \in E_1^*$ via Hahn-Banach.

i.e. $g(\lambda y_0) = \lambda$ (define g on span$\{y_0\}$)

then extend g to all of E. $\Rightarrow \|g\|_{E_1^*} = 1$

Note: $g(y_0) = 1$ and $g(Ax_0) = \|Ax_0\|_{E_1}$

Now pull this g back using A^*.

$$\|Ax_0\|_{E_1} = |g(Ax_0)| = |A^*g(x_0)|$$

$$\leq \|A^*g\|_{E^*} \|x_0\|_E$$

$$\leq \|A^*\|_{\mathcal{L}(E_1^*, E^*)} \|g\|_{E_1^*} \|x_0\|_E$$

$$= \|A^*\|_{\mathcal{L}(E_1^*, E^*)} \|x_0\|_E$$

since $\|g\|_{E_1^*} = 1$

(note! Here we used that A^* is a bounded linear op. from E_1^* to E^*)

$$\Rightarrow \quad \|Ax_0\|_{E_1} \leq \|A^*\|_{\mathcal{L}(E_1^*, E^*)} \|x_0\|_E$$

Now let x_0 vary.

$\Rightarrow$ We've just shown

$$\|A\|_{\mathcal{L}(E,E_1)} \leq \|A^*\|_{\mathcal{L}(E_1^*, E^*)}.$$

Combining the two inequalities,

$$\|A\|_{\mathcal{L}(E,E_1)} = \|A^*\|_{\mathcal{L}(E_1^*, E^*)}.$$

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Now return to Hilbert spaces.

We know that if $\phi \in H^*$ then $\exists!\, y \in H$ so that $\phi(x) = \langle x, y\rangle \;\; \forall x \in H$.

$T : H \to H^*$ where $T(y) = \langle \cdot, y\rangle$ is an isomorphism between H and H^*.

let $A : H \to H$ be a linear operator then $A^* : H^* \to H^*$ is its adjoint

and

$$\begin{tikzcd} H \arrow[d, "T"'] & H \\ H^* \arrow[r, "A^*"'] & H^* \arrow[u, "T^{-1}"'] \end{tikzcd}$$

is a bounded linear operator from H to H. call it $\tilde{A}^*$

$$\Rightarrow \tilde{A}^* = T^{-1} A^* T \Rightarrow T\tilde{A}^* = A^* T$$

let $y \in H$ then $A^* T(y)(x) = T(y)(Ax)$
$= \langle Ax, y\rangle$

on the other hand $T\tilde{A}^*(y)(x) = \langle x, \tilde{A}^* y\rangle$

$$\Rightarrow \langle Ax, y\rangle = \langle x, \tilde{A}^* y\rangle \quad \forall x \in H, \forall y \in H$$

In this way given $A^* : H^* \to H^*$ we define a unique $\tilde{A}^* : H \to H$. In an abuse of notation, $\tilde{A}^* = A^*$

and $\langle Ax, y\rangle = \langle x, A^* y\rangle \;\; \forall x, y \quad A^* : H \to H$.

If $A: H \to H$ where H is a Hilbert space and A is bounded + linear, this makes us ask if $A^*: H \to H$ is the same as A.

<u>defn</u>: if H is a Hilbert space and $A: H \to H$ is a bounded linear operator then A is <u>self-adjoint</u> if

$$\langle Ax, y \rangle = \langle x, Ay \rangle \quad \forall x, y \in H.$$

Ooooh... Now we can begin to ask questions about eigen values!

Recall if $A: \mathbb{R}^n \to \mathbb{R}^n$ and A is symmetric ($A^T = A$) then A has a full set of eigenvalues (n of them.)

But what happens in infinite dimensions?

First, recall the finite dimensional case

$A: H \to H$ then $\lambda \in \mathbb{C}$ is an eigenvalue of A if the equation

$$Ax = \lambda x$$

has at least one nonzero solution. x is called an eigenvector of A. (I'm taking H a complex vector space here.)

then we say the set of all eigenvalues of A is the spectrum and all other values of λ are regular.

λ is regular $\Leftrightarrow$ $(A - \lambda I)$ is invertible

and in this case $(A-\lambda I)^{-1}$ is automatically bounded since a linear operator on a finite dimensional vector space is automatically bounded.

i.e. In *finite* dimensions there are 2 possibilities

1) $\exists$ nonzero x so that $Ax = \lambda x$
(i.e. λ is eigenvalue and $(A-\lambda I)^{-1} \nexists$

2) $(A-\lambda I)^{-1}$ exists and is bounded
(i.e. λ is a regular point)

In infinite dimensions, there's a third possibility:

3) $(A-\lambda I)^{-1}$ exists <u>but</u> <u>is</u> <u>not</u> <u>bounded</u>.

<u>Rule 1</u>: In infinite dimensions we consider <u>complex</u> topological vector spaces to avoid the $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ problem

New definitions for you to learn:

given $A : E \to E$

the operator $R_\lambda = (A - \lambda I)^{-1}$ is called <u>the resolvent of A</u>.

λ is a <u>regular point of A</u> if $R_\lambda(x)$ is defined for <u>all</u> $x \in E$ and is continuous.

λ is in the <u>spectrum of A</u> if λ is not a regular point.

spectrum of A = (point spectrum of A) $\cup$ (continuous spectrum of A)

λ is in the <u>point spectrum of A</u> if $R_\lambda(x)$ cannot be defined $\forall x \in E$

λ is in the <u>continuous spectrum of A</u> if R_λ can be defined on all E but it is not a <u>continuous</u> linear operator.

<u>Q</u>: When do we only have a point spectrum?

<u>Q</u>: Even if we only have a point spectrum, is that helpful?

<u>Q</u>: What does it mean to have a continuous spectrum?

theorem: Let $A: E \to E$ where E is a Banach space then the spectrum of A is closed.

proof: It suffices to show that the set of regular points is open. Let λ be a regular point.

$\Rightarrow (A-\lambda I)^{-1}$ exists and is bounded.

Choose $\delta < \frac{1}{\|(A-\lambda I)^{-1}\|}$. Consider $-\delta I : E \to E$

then $\|-\delta I\| < \frac{1}{\|(A-\lambda I)^{-1}\|}$

$\Rightarrow$ Since E is complete

$(A-\lambda I) + (-\delta I)$ is invertible

and its inverse is a bounded linear operator. $\Rightarrow$ $\delta + \lambda$ is a regular point for all $|\delta| < \frac{1}{\|(A-\lambda I)^{-1}\|}$

$\Rightarrow$ the set of regular points is open $\Rightarrow$ Spectrum is closed //

theorem: if A is a bounded linear operator $A: E \to E$ where E is a Banach space and

$|\lambda| \geq \|A\|$ then λ is a regular point i.e. if A is a bounded linear operator then spectrum$(A) \subseteq \mathbb{C}$ is a bounded set

proof: $A - \lambda I = -\lambda\left[I - \frac{A}{\lambda}\right]$

and if we can invert $I - \frac{A}{\lambda}$ w/ $(I - A/\lambda)^{-1}$ being a continuous linear operator then we have $(A-\lambda I)^{-1}$ exists and is continuous.

We know $\|-A/\lambda\| = \frac{\|A\|}{|\lambda|} < 1 = \frac{1}{\|I^{-1}\|}$

and since E is a Banach space, this implies $\left(I - \frac{A}{\lambda}\right)^{-1}$ exists and is a continuous linear operator $\Rightarrow \lambda$ is a regular point. //

theorem: $A: H \to H$, A is a self adjoint operator, H a complex Hilbert space. Then 1) $\lambda \in$ point spectrum $\Rightarrow \lambda \in \mathbb{R}$ and 2) eigenvectors of distinct point spectrum values are orthogonal.

That's just like what we learnt in linear algebra... it's just inner product games...

proof: if $Ax = \lambda x$ $x \neq 0$ then

$$\lambda \langle x, x\rangle = \langle \lambda x, x\rangle = \langle Ax, x\rangle$$

$$= \langle x, Ax\rangle \quad \text{since } A \text{ selfadj.}$$

$$= \langle x, \lambda x\rangle$$

$$= \bar{\lambda}\langle x, x\rangle$$

$\vec{x} \neq 0 \Rightarrow$ divide out $\langle x, x\rangle \Rightarrow \lambda = \bar{\lambda}$. ✓

assume $Ax = \lambda x$ $\quad Ay = \mu y \quad x \neq 0, y \neq 0, \lambda \neq \mu$

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Ax, y\rangle$$

$$= \langle x, Ay\rangle$$

$$= \langle x, \mu y\rangle = \bar{\mu}\langle x, y\rangle = \mu\langle x, y\rangle$$

$$(\lambda - \mu)\langle x, y\rangle = 0$$

$$\Rightarrow \lambda \neq \mu \Rightarrow \langle x, y\rangle = 0 \ ✓,$$

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Q: Give me some bounded linear operators w/ interesting spectra... interesting ramifications...