Baire Category Theorem, Open Mapping Theorem

Let's go back to topological ~~ve~~ spaces for a while. We don't have a metric, we just have a topology. Is there some sense of a "small" set or a "large" set when you don't have a metric? (or a measure?)

Recall: Let $A \subset X$ then $x \in A$ is in the <u>interior</u> of A if $\exists\, U \in \tau$ so that $x \in U \subseteq A$.

$\mathring{A} :=$ the set of interior points of A.

(Note: $\mathring{A}$ can be $\emptyset$ even though A is dense in X. Take $(X, \tau) = (\mathbb{R}$ w/ usual metric topology. Take $A = \mathbb{Q}$. Then $[A] = \mathbb{R}$, $\mathring{A} = \emptyset$.)

<u>defn:</u> $A \subseteq X$ is <u>nowhere dense</u> if its closure has empty interior: $[A]^\circ = \emptyset$.

($\mathbb{Q}$ is dense but it's not nowhere dense!)

<u>defn:</u> If A is a countable union of nowhere dense sets then A is <u>meager</u> or a <u>set of first category</u>. (or "a set of first Baire 'category'").

<u>defn:</u> A is a <u>residual set</u> if $X - A$ is meager.

<u>defn</u>: $A \subseteq X$ is <u>dense in X</u> if $[A] = X$.
If $A \subseteq B$ then A is <u>dense in B</u> if $B \subseteq [A]$.

<u>defn</u>: A is a <u>set of second Baire category</u> if A is not a set of first Baire category. (Well, that's a definition to make a mathematician proud! :))

<u>fact</u>: A is nowhere dense if and only if $A^c = X - A$ has dense interior

<u>fact</u>: A is dense if and only if its complement has empty interior

Very roughly:

"very small" ~ nowhere dense
"very large" ~ dense interior

"small" ~ meager
"large" ~ residual

"medium small" ~ empty interior
"medium large" ~ dense

ex: $A = \{(x,y) \in \mathbb{R}^2 \mid y = x^2\}$. Then A is closed in $\mathbb{R}^2$, $A^\circ = \emptyset$. $\Rightarrow$ A is nowhere dense in $\mathbb{R}^2$

ex: Let $\{x_n\}_1^\infty \subset \mathbb{R}$. Then this sequence is a set of first category (is meager) since it's the union of singletons $\{x_i\}$ and singletons are nowhere dense.

ex: $\mathbb{Q} \subseteq \mathbb{R}$

is meagre (since countable) ~ "small"
is dense ~ "medium large"
has empty interior ~ "medium small"

so the zoology is not totally useful...

ex: the irrationals are residual since their complement is meagre. Q: does this mean they're not meagre? Is it possible for a set to be both meagre and residual? Yes!

Baire Category Theorem $\Rightarrow$ Irrationals are not meagre.

theorem. $A \subset X$ is nowhere dense if and only if given any nonempty open set V, $A \cap V$ is not dense in V.

(this is why we say "nowhere dense").

proof:

($\Rightarrow$) Assume not. i.e. $\exists\ U \in \mathcal{T}$, $U \neq \phi$ so that $A \cap U$ is dense in U.

i.e. $U \subseteq [A \cap U]$. $\Rightarrow U \subseteq [A]$.

Since $U \neq \phi$ and U is open, this shows that $[A]^\circ \neq \phi$. $\Rightarrow A$ is not nowhere dense.

($\Leftarrow$) Assume not. i.e. A is not nowhere dense.

$\Rightarrow [A]^\circ \neq \phi$. Let $x \in [A]^\circ$. $\Rightarrow \exists$ open set V with $x \in V \subseteq [A]$. I want to show that this implies $V \subseteq [A \cap V]$. (this would then contradict the second part since $V \neq \phi$ and we're done.) Let $y \in V$.

Let U be a nbd of y. Since V is open, $U \cap V$ is also a nbd of y. Since $y \in [A]$, this means that $(U \cap V) \cap A \neq \phi$. $\Rightarrow U \cap (A \cap V) \neq \phi$.

Since U was arbitrary, this shows $y \in [A \cap V]$.

This proves $V \subseteq [A \cap V]$ and we're done!

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fact: a residual set can be written as a countable intersection of sets w/ dense interiors

fact: the countable union of nowhere dense sets can be

a) dense (write $\mathbb{Q}$ as the countable union of singletons)
b) nowhere dense (write $\mathbb{N}$ as countable union of single)
c) neither nowhere dense nor dense
(write $\mathbb{Q} \cap [0, \infty)$ as countable union of singletons.)

fact: finite union of nowhere dense sets is nowhere dense

fact: let $f: X \to Y$ be continuous. Can we say anything like

$$f^{-1}(\text{dense}) = \text{dense}$$
$$f^{-1}(\text{nowhere dense}) = \text{nowhere dense?}$$

No! take $f(x) = y_0$ where y_0 is fixed. (i.e. f the constant function.) Assume a reasonable topology on Y. Then

$\{y_0\}$ is nowhere dense
$Y - \{y_0\}$ is dense.

but $f^{-1}(\{y_0\}) = f^{-1}(\text{nowhere dense}) = X$ ← not nowhere dense!
$f^{-1}(Y - \{y_0\}) = f^{-1}(\text{dense}) = \emptyset$ ← not dense!

fact. A set can be both meagre and residual.

example?

let $(X, \tau) =$ $\mathbb{Q}$ w/ metric topology.

let $A = X - \{1/2\}$.

A is meagre since it's a countable union of nowhere dense sets.

A is residual since $A^c = \{1/2\}$ is nowhere dense (and is therefore meagre)

fact: let f be a continuous real-valued function on $\mathbb{R}$. let $A = \{x \mid f(x) = 0\}$

A is nowhere dense $\iff$ there is no interval (a, b) on which f is identically 0.

fact: the cantor set is nowhere dense in $[0, 1]$.

Baire Category Theorem In a complete metric space every residual set is dense.

proof: recall that a set is residual if it can be written as the countable intersection of sets with dense interiors.

let A be our residual set.

$A = \bigcap_1^\infty U_n$ where $(U_n)^\circ$ is dense.

I want to show that A is dense. i.e. if $V \in \mathcal{T}$, $V \neq \emptyset$ $\bigcap_1^\infty U_n$ intersects V.

Since $A = \bigcap_1^\infty U_n \supseteq \bigcap_1^\infty (U_n)^\circ$, it suffices to show that $\bigcap_1^\infty (U_n)^\circ$ intersects V.

Since U_1° is dense, $U_1^\circ \cap V \neq \emptyset \Rightarrow \exists\, x_1 \in U_1^\circ \cap V$.

$U_1^\circ \cap V$ is open $\Rightarrow \exists$ ball of radius $< 1 \ni S(x_1, r_1) \subset U_1^\circ \cap V$.

Since U_2° is dense and $S(x_1, r_1)$ is open, $\exists\, x_2 \in U_2^\circ \cap S(x_1, r_1)$

$\Rightarrow x_2 \in U_2^\circ \cap U_1^\circ \cap V$. Choose radius $< 1/2 \ni$ $S(x_2, r_2) \subset U_2^\circ \cap U_1^\circ \cap V$. Etc. In this way, we construct a nested sequence of balls so that

$$S(x_n, r_n) \subset \left(\bigcap_1^n U_i^\circ\right) \cap V$$

Since $m \geq n \Rightarrow S(x_m, r_m) \subset S(x_n, r_n) \Rightarrow d(x_m, x_n) < \frac{2}{n}$

$\Rightarrow \{x_n\}$ is a cauchy sequence. X is complete

$\Rightarrow \exists\, x_\infty \in X$ with $x_n \rightarrow x_\infty$

$$\Rightarrow x_\infty \in \bigcap_1^\infty B_n \subset \left(\bigcap_1^\infty U_n^\circ\right) \cap V \subseteq A \cap V$$

$\Rightarrow A \cap V \neq \emptyset$ and A is dense in X! //

Corr. A complete nonempty metric space cannot be the countable union of nowhere dense sets.
(It cannot be a meagre set.)

proof: Assume (X, ρ) is complete and meagre.
Then X^c is residual, $\Rightarrow$ BCT says X^c is dense. $\Rightarrow$ $\emptyset$ is dense in X $\times$

recall G_δ and F_σ

A is G_δ if A is countable intersection of open sets
A is F_σ if A is countable union of closed sets

corr: consider $\mathbb{Q}$ in $\mathbb{R}$ w/ usual metric topology.
then $\mathbb{Q}$ is not a G_δ set.

proof: Assume not $\mathbb{Q} = \bigcap_1^\infty U_n$ where U_n is open.
$\Rightarrow \mathbb{Q} \subseteq U_n$ each n. $\Rightarrow \mathbb{R} = [\mathbb{Q}] = [U_n]$ each n.
$\Rightarrow$ each U_n is dense in $\mathbb{R}$.

$\Rightarrow \mathbb{Q}^c = \bigcup_1^\infty (U_n)^c$ $\qquad (U_n)^c$ is nowhere dense
(a set is nowhere dense $\Leftrightarrow$ its complement has dense interior)

And we already know
$\mathbb{Q}$ is meagre $\Rightarrow \mathbb{Q} = \bigcup_1^\infty V_n$ where
V_n is nowhere dense $\Rightarrow \mathbb{R} = \bigcup_1^\infty V_n \cup \bigcup_1^\infty (U_n)^c \Rightarrow \mathbb{R}$ is meagre. $\times$

<u>Defn:</u> Let $(X, \mathcal{T})$ be a topological space. A property P that elements of X may or may not have is <u>generic</u> <u>(in the Baire sense)</u> if

$$\{x \mid x \text{ has property } P\}$$

is a dense residual set.

(Note: this allows you to say "for almost every" when you don't have a measure on X!)

<u>ex:</u> in $\mathbb{R}$ w/ metric topology, the property of being irrational is generic.

<u>ex:</u> X = continuous functions on $[0,1]$ w/ L^∞ metric topology.

Then differentiable functions are meagre. Further, being nondifferentiable is generic.

<u>ex:</u> $X = \mathbb{C}^n$ w/ Zariski topology

$(a_1 \ldots a_n)$ has property P if the polynomial

$$p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n$$

has n <u>distinct</u> roots in $\mathbb{C}$. This property is <u>generic</u>.

<u>ex:</u> the generic $n \times n$ complex matrix has distinct eigenvalues.

Okay, okay on to the open mapping theorem!

Theorem: Let X and U be Banach spaces and $M: X \to U$ a bounded linear operator onto U. Then $\exists\, \delta > 0$ so that the image of the open unit ball in X under M contains the ball of radius δ in U:

$$S(\vec{0}, \delta) \subseteq M\, S(\vec{0}, 1).$$

proof: Let B_n denote the open ball of radius n around $\vec{0}$ in either space X or U.

Since M maps X onto U and since $X = \bigcup_1^\infty B_n$, it follows that $U = \bigcup_1^\infty M B_n$. Since U is a complete metric space (metric inherited from norm) the Baire category theorem tells us that U cannot be the countable union of nowhere dense sets.

$\Rightarrow$ One of the MB_n is not nowhere dense.

$\Rightarrow$ $\exists$ an open set $V \subseteq U$ so that $MB_n \cap V$ is dense in V. Translating V and MB_n, we have that some translate of MB_n is dense in some ball around $\vec{0}$ (in U.)

Since the range of M is all of U, we can write that translate as $M(B_n - x_0)$ for some $x_0 \in X$.

$\Rightarrow$ $M(B_n - x_0)$ is dense in some open ball around $\vec{0}$.

On the other hand $B_n - x_0$ is contained in the ball of radius $n + \|x_0\|_X$ around $\vec{0}$. $\Rightarrow$ if we take $m > n + \|x_0\|_X$ we have $M B_m$ is dense in some open ball around $\vec{0}$. Now, by the homogeneity of M we conclude $M B_1$ is dense in B_r for some $r > 0$. And for any $c > 0$, $M B_c$ is dense in B_{cr}.

We want to show that any point $u \in B_r$ is the image of some $x \in B_2$. (this will then show $B_r \subseteq M B_2$ and by homogeneity $B_{r/2} \subseteq M B_1$ and we're done!) We construct the point $x \in B_2$ as an infinite series $x = \sum_1^\infty x_j$.

First, let x_1 satisfy $\|x_1\| < 1$ <u>and</u> $\|u - Mx_1\| < r/2$ (can do this since MB_1 dense in B_r.)

Now we choose x_2 so that

$$\|x_2\| < 1/2 \quad \underline{\text{and}} \quad \|u - Mx_1 - Mx_2\| < r/4$$

Why can we do that? We know

$$MB_{1/2} \text{ is dense in } B_{r/2}.$$

Since $\|u - Mx_1\| < r/2$ we know $u - Mx_1 \in B_{r/2}$.

$\Rightarrow \exists\, x_2 \in B_{1/2} \ni \| (u - Mx_1) - Mx_2\| < r/4.$

Proceeding inductively, we choose x_n so that $\|x_n\| < \frac{1}{2^{n-1}}$ and $\left\|u - \sum_1^n Mx_k\right\| < \frac{r}{2^n}$

Since $MB_{\frac{1}{2^{n-1}}}$ is dense in $B_{r/2^{n-1}}$.

Now a normed vector space is complete
$\iff \sum_1^\infty \|x_n\| < \infty \Rightarrow \sum_1^\infty x_k$ converges.

And $\sum_1^\infty \|x_n\| < \infty$ and X is complete $\Rightarrow \sum_1^\infty x_k = x_\infty \in X$

By construction, $\|x\| \le \sum_1^\infty \|x_n\| < \sum_1^\infty \frac{1}{2^{n-1}} = 2.$

And, since M is a bounded linear map and

$y_n = \sum_1^n x_k$ has $y_n \to x_\infty$ we have $My_n \to Mx_\infty$

and $My_n \to u$ by constr. $\Rightarrow u = Mx_\infty$ and done! //

Corr: Let X and U be Banach spaces and $M: X \to U$ a bounded linear operator. Then M is an open mapping: $M(\text{open}) = \text{open}$.

Corr: Let X and U be Banach spaces and $M: X \to U$ be a bounded linear operator that is 1:1 and onto. Then the inverse of M is a bounded linear map from U to X:

(recall, we used this in the proof of the closed graph theorem...)

proof: from the open mapping theorem $\exists\ \delta > 0$ so that $B_\delta \subseteq M B_1$. $\Rightarrow$ if $y \in U$ $\|y\| = \frac{\delta}{2}$ then $\exists\ x$ w/ $\|x\| \leq 1$ and $Mx = y$.

i.e. $\|x\| \leq 1 = \dfrac{2\|y\|}{\delta}$. By homogeneity of M for every $y \in U$ $\exists$ $x \in X$ so that

$$Mx = y \text{ and } \|x\| \leq \frac{2\|y\|}{\delta}$$

Now, since M is 1:1, $x = M^{-1}y$.

$\Rightarrow \quad \|M^{-1}y\| \leq \frac{2}{\delta}\|y\| \quad \forall\, y \in U.$

$\Rightarrow$ M^{-1} is a bounded linear functional (check linearity!) and

$$\|M^{-1}\|_{\mathcal{L}(U,X)} \leq \frac{2}{\delta}.$$

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