theorem:

let E, E_1 be Banach spaces and $\{A_n\} \subseteq \mathcal{L}(E, E_1)$ a sequence of maps that is

1) uniformly bounded,

$$\|A_n\|_{\mathcal{L}(E, E_1)} \leq C < \infty \qquad \forall n$$

2) pointwise convergent on a dense set in E

$$\lim_{n \to \infty} A_n x \quad \text{exists} \quad \forall\, x \in \text{dense set}$$

then $\exists\, A \in \mathcal{L}(E, E_1)$ so that $A_n \to A$ in the $\mathcal{L}(E, E_1)$ norm (i.e. $A_n \to A$)

proof: Reread the proof of the first theorem from class on Dec 2, 2002. Mimic that proof's steps 1-5. //

recall for continuous linear real-valued functionals, we had the following theorem:

assume $(E, \|\ \|)$ is complete (a Banach space) and $\{\phi_n\} \subseteq E^*$ has $\phi_n \underset{w*}{\rightharpoonup} \phi$. Then $\{\phi_n\}$ is strongly bounded: $\exists\, C < \infty$ such that $\|\phi_n\|_{E^*} \leq C \quad \forall n$.

i.e. weak* convergence $\Rightarrow$ strongly bounded.
(or uniformly bounded).

<u>Q:</u> Does this generalize to bounded linear operators $E \to E_1$? Yes!

<u>defn:</u> $\{A_n\} \subseteq \mathcal{L}(E, E_1)$ is weak convergent if for each $x \in E$, $A_n x \underset{w}{\to} y \in E_1$ for some $y \in E_1$.
(i.e $\{A_n x\}$ is w convergent in E_1 for each $x \in E_1$.)

<u>Note:</u> if $\{A_n\}$ is weak convergent then you can check that $Ax :=$ weak limit of $A_n x$ defines $A \in \mathcal{L}(E, E_1)$.

<u>Note:</u> you can check that if $E_1 = \mathbb{R}$ (or $\mathbb{C}$)
then $\{A_n\} \subset E^*$. Further, you can check that $\{A_n\}$ weak convergent in $\mathcal{L}(E, E_1)$ is the same as A_n is weak* convergent in E^*

<u>theorem:</u> Let E and E_1 be Banach spaces, $\{A_n\} \subseteq \mathcal{L}(E, E_1)$ and for each $x \in E$ and each $\phi \in E_1^*$ $\exists\, C_{x,\phi} < \infty$ so that

$$|\phi(A_n x)| \leq C_{x,\phi} \qquad \forall n$$

<u>then</u> $\{A_n\}$ is uniformly bounded. i.e $\exists\, C < \infty$ so that

$$\|A_n\|_{\mathcal{L}(E, E_1)} \leq C \qquad \forall n.$$

<u>Corr:</u> If $\{A_n\}$ is weak convergent then $\{A_n\}$ is uniformly bounded.

proof of uniform boundedness theorem:

First, recall the uniform boundedness theorem for linear functionals:

if $\{y_n\} \subseteq E_1$ is a sequence such that for all $\phi \in (E_1)^*$ $|\phi(y_n)| \le C_\phi < \infty$ $\forall n$

then $\exists$ C so that $\|y_n\|_{E_1} \le C < \infty$ $\forall n$. (i.e. if $\{y_n\}$ is weak $*$ bounded then $\{y_n\}$ is strongly bounded.)

Fix $x \in E$. then $A_n x$ is a sequence in E_1. And by assumption, given $\phi \in (E_1)^*$ $\exists$ $C_{x,\phi}$ $\ni$ $|\phi(A_n x)| \le C_{x,\phi}$ $\forall n$. $\Rightarrow$ $\exists C_x < \infty$ so that $\|A_n x\|_{E_1} \le C_x$ $\forall n$.

Recall the principle of uniform boundedness for metric spaces:

If a collection $\{f_n\}$ of continuous real-valued functions on a complete metric space is bounded at each point:

$$|f_n(x)| \le M_x < \infty \qquad \forall n$$

then the functions are uniformly bounded on some nonempty open set U:

$$|f_n(x)| \le M < \infty \ \forall n, \ \forall \ x \in U$$

We're in this situation!

$$f_n : \quad x \longrightarrow \|A_n x\|_{E_1}$$

$\{f_n\}$ is a family of continuous real-valued functions that is bounded at every point x of a complete metric space.

Moreover, f_n is linear and convex, something we'll use momentarily. By the principle of uniform boundedness that I just quoted, $\exists$ an open Ball in E such that

$$|f_n(x)| \leq M < \infty \quad \forall x \in \text{open ball.}$$

i.e. $\exists$ $z \in E$ and $r > 0$ so that $\|x - z\|_E < r$

$$\Rightarrow |f_n(x)| = \|A_n x\|_{E_1} \leq M.$$

Specifically, let $y \in E$ with $\| \ \|_E = r/2$. Then $y = x - z$ for some x in ball of radius r around z

therefore

$$\begin{aligned} |f_n(y)| = |f_n(x-z)| &= \|A_n(x-z)\|_{E_1} \\ &= \|A_n x - A_n z\|_{E_1} \\ &\leq \|A_n x\|_{E_1} + \|A_n z\|_{E_1} \\ &= |f_n(x)| + |f_n(z)| \leq 2M \end{aligned}$$

since x, z are both in ball of radius r around z

Let x_0 be an arbitrary point in E.
then $y = \frac{r}{2} \frac{x_0}{\|x_0\|_E}$ satisfies $\|y\|_E = r/2$

$\Rightarrow$ By previous argument, $|f_n(y)| \le 2M$

i.e. $\|A_n y\|_{E_1} \le 2M$

$$\Rightarrow \left\| A_n\left(\frac{r}{2} \frac{x_0}{\|x_0\|_E} \right) \right\|_{E_1} \le 2M$$

$$\Rightarrow \frac{r}{2\|x_0\|_E} \|A_n x_0\|_{E_1} \le 2M$$

$$\Rightarrow \|A_n x_0\|_{E_1} \le \frac{4M}{r} \|x_0\|_E \quad \forall\, x_0 \in E$$

$$\Rightarrow \|A_n\|_{\mathcal{L}(E,E_1)} \le \frac{4M}{r}$$ as desired! //

The Closed Graph Theorem

defn. Let E and E_1 be Banach spaces.
then $A : E \to E_1$ is closed if whenever

1) $x_n \to x$ in E and 2) $Ax_n \to y$ in E_1

then $Ax = y$.

fact: if A is continuous then A is closed.

Theorem: Let E and E_1 be Banach spaces and $A: E \to E_1$ a linear operator.
if A is a closed mapping then A is continuous.

This is the closed graph theorem. It has some surprising applications

proof of theorem: Define a new vector space
$G = \{ (x, Ax) \mid x \in E \}$ Addition and scalar multiplication defined in the obvious manner. We put a norm on G by

$$\| (x, Ax) \|_G := \|x\|_E + \|Ax\|_{E_1}$$

This is definitely a norm. I claim that G is complete wrt this norm: Let $\{(x_n, Ax_n)\}$ be Cauchy in G. Then given $\varepsilon > 0$ $\exists$ $N_\varepsilon \ni$ $m, n \geq N_\varepsilon \Rightarrow$

$$\| (x_m - x_n, Ax_n - Ax_m) \|_G < \varepsilon$$

i.e. $\|x_m - x_n\|_E + \|Ax_n - Ax_m\|_{E_1} < \varepsilon$

this implies $\{x_n\}$ is Cauchy in E and $\{Ax_n\}$ is Cauchy in E_1. But E and E_1 are complete $\Rightarrow$ $x_n \to x$ and $Ax_n \to y$. Now since A is <u>closed</u> we know $Ax = y$.

this proves $(x_n, Ax_n) \longrightarrow (x, Ax) \in G$ $\Rightarrow$ G complete with respect to $\|\cdot\|_G$.

We now define $P : G \to E$ by $P(g)$ x.

then $\|Pg\|_E = \|x\|_E \le \|(x, Ax)\|_G = \|g\|_G$

$\Rightarrow$ P is a bounded linear operator from G to E.

Furthermore P maps G onto E and P is $1:1$.

$\Rightarrow$ P^{-1} is bounded. (why?!? we'll come back to this.) Since P^{-1} is bounded, $P^{-1}: E \to G$, $\exists\, c < \infty$ s.t. that

$$\|P^{-1}x\|_G \le c\|x\|_E \quad \forall x \in E.$$

but $x \in E \Rightarrow x = Pg$

$\Rightarrow \|g\|_G \le C\,\|Pg\|_E \quad \forall g \in G$

$\Rightarrow \|x\|_E + \|Ax\|_{E_1} = \|g\|_G \le C\|x\|_E \Rightarrow \|Ax\|_{E_1} \le (C-1)\|x\|_E$

$\forall x \in E$ $\Rightarrow$ A is continuous. //

Note: G is the graph of A. And requiring A to be closed is the same as requiring that G be closed in $E \times E_1$.

Note: P bounded, P 1:1, P onto $\Rightarrow$ P^{-1} bounded is nonobvious. It's a corollary of the open mapping theorem, which we'll prove next time.

Theorem: Let E be a vector space and let $\|\cdot\|_1$ and $\|\ \|_2$ be two norms on E. Assume the norms are compatible: if $\|x_n - x\|_1 \to 0$ and $\|x_n - y\|_2 \to 0$ then $x = y$.

Assume $(E, \|\cdot\|_1)$ is a Banach space and $(E, \|\cdot\|_2)$ is a Banach space. Then $\|\cdot\|_1$ and $\|\ \|_2$ are equivalent.

Proof: Let $(E_1, \|\cdot\|) := (E, \|\cdot\|_1)$ and $(E_2, \|\cdot\|) := (E, \|\cdot\|_2)$. By assumption, E_1 and E_2 are Banach spaces. Since $\|\cdot\|_1$ and $\|\ \|_2$ are compatible, $Id : E_1 \to E_2$ is a closed mapping. $\Rightarrow$ By CGT, $\exists\, c_1 \ni$ $\|Id\, x\|_{E_2} \leq c_1 \|x\|_{E_1}$ i.e. $\|x\|_2 \leq c_1 \|x\|_1$. Now reverse roles of E_1 and E_2. done!

Theorem: Let E and E_1 be Banach spaces and $A: E \to E_1$ a bounded linear map. Assume that $A(E)$ (the range of A) has finite codimension in E_1. Then $A(E)$ is closed.

proof: exercise. Hint: extend A so that $A: E \oplus \mathbb{R}^n \to E_1$ is linear, bounded, and <u>onto</u> E_1 (choose ~~the~~ n appropr.)

Theorem: Let E be a Banach space and $Y \subseteq E$ and $Z \subseteq E$ be (closed) subspaces so that $E = Y \oplus Z$. (ie every $x \in E$ can be uniquely written as $y + z$) Denote

$$P_Y : E \to Y \qquad P_Y(x) = y$$

$$P_Z : E \to Z \qquad P_Z(x) = z$$

then

1) P_Y is linear and onto Y, P_Z is linear and onto Z
2) $P_Y \circ P_Y = P_Y \qquad P_Z \circ P_Z = P_Z \qquad P_Y \circ P_Z = 0$
3) P_Y and P_Z are continuous.

Proof: 1) and 2) are obvious. 3) follows directly from the CGT (since we assumed Y and Z closed). done! //