Linear Operators

We've just done a fair bit of work on continuous linear functionals (linear mappings from the vector space to the field).

Now, we consider linear mappings between vector spaces.

Let E and E_1 be two topological vector spaces. Let $D_A \subseteq E$ be a subspace and

$$A : D_A \longrightarrow E_1$$

where A is linear $A(\alpha x_1 + \beta x_2) = \alpha A x_1 + \beta A x_2$

$\forall \alpha, \beta \quad \forall x_1, x_2 \in D_A$.

defn: let $x_0 \in D_A$. Then A is continuous at x_0 if given V a neighborhood of Ax_0, $\exists$ neighborhood U of x_0 so that $A(U \cap D_A) \subseteq V$. (i.e. $A^{-1}(V)$ is open in the relative topology on D_A.)

defn: A is continuous if it is continuous at every point of D_A

<u>Note:</u> if E and E_1 are both <u>normed</u> vector spaces then A is continuous if and only if given $\varepsilon > 0$ $\exists\, \delta > 0$ so that

$$\|x' - x''\| < \delta \Rightarrow \|Ax' - Ax''\| < \varepsilon \quad \forall x', x'' \in D_A$$

<u>defn:</u> A linear operator $A: E \to E_1$ is <u>bounded</u> if $M \subseteq E$, M bounded $\Rightarrow AM$ is bounded in E_1.

<u>theorem:</u> Let (E, τ) be a topological vector space. $A: E \to E_1$ linear operator.

1) if A is continuous then A is bounded
2) if (E, τ) satisfies first axiom of countability then A bounded $\Rightarrow A$ continuous.

<u>corr:</u> if (E, τ) is a <u>normed</u> vector space then A bounded $\Leftrightarrow A$ continuous.

Before proving the theorem, some examples of continuous linear operators ex 1: $Id : E \to E$

$Id(x) = x$ the identity operator

ex 2: $O : E \to E_1$

$O(x) = \vec{O} \quad \forall x \in E$ the zero operator

ex 3: $E = \mathbb{R}^m \quad E_1 = \mathbb{R}^n$

then if we fix coords in $\mathbb{R}^m$ and $\mathbb{R}^n$

$$A(x) = A\left(\sum_1^m x_j e_j\right) = \sum_1^m x_j A e_j$$

$$A e_j \in \mathbb{R}^n \Rightarrow A e_j = \sum_1^n a_{ij} e_i'$$

$$\Rightarrow A(x) = \sum_1^m x_j \sum_1^n a_{ij} e_i' = \sum_1^n y_i e_i'$$

where $y_i = \sum_1^m a_{ij} x_j$

(ooh! Matrices!)

ex: Let H be a Hilbert space H_1 a subspace

then $H = H_1 \oplus (H_1)^\perp$

define $\pi_1 : H \to H_1$ by $\pi_1(x+y) = x$

(This one is a little trickier... We'll return to it...)

What are some more interesting bounded operators?

Let $L = C([a,b])$ and recall our norms on L:
$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$$
$$\|f\|_p = \sqrt[p]{\int_a^b |f(x)|^p dx}$$

let $L^\infty := (L, \|\cdot\|_\infty)$ $\quad L^p := (L, \|\cdot\|_p)$

ex: The Fourier transform. In this case $[a,b] = (-\infty, \infty)$
$$(Ff)(x) = \frac{1}{\sqrt{2\pi}} \int f(t) e^{-ixt} dx$$

then $F : L^1 \to L^\infty$ is bounded operator

$F : L^2 \to L^2$ is a bounded operator

combining these two facts (oooh... magic)

$F : L^p \to L^{\frac{p}{p-1}}$ is bounded operator for all $p \in [1,2]$

ex: the Hilbert transform. Again, in this case $[a,b] = (-\infty, \infty)$
$$\mathcal{H}h(\tau) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{h(t)}{t-\tau} dt$$

fact: $\mathcal{H} : L^p \to L^p$ is a bounded operator $\forall p \in (1, \infty)$

fact: $\mathcal{H} : L^2 \to L^2$ is an isometry

$$\|\mathcal{H} h\|_{L^2} = \|h\|_{L^2}$$

ex: the Laplace transform. In this case, $[a,b] = [0, \infty)$

$$\mathcal{L}f(s) = \int_0^\infty f(t) e^{-st} dt$$

fact: $\mathcal{L} : L^2 \to L^2$ is a bounded operator

fact: $\mathcal{L} : L^p \to L^p$ is not bounded if $p \in (1, \infty)$, $p \neq 2$

Plus, there are lots of cool examples of bounded operators that arise in PDE.

Right. Let's return to the proof of the theorem relating continuity of A to boundedness of A

Proof:

($\Longrightarrow$) Assume Not. i.e. Assume $\exists$ bounded set $M \subseteq E$ so that AM is unbounded in E_1. I'll then prove that A is not continuous.

Since AM is unbounded, $\exists$ a neighborhood V of $\vec{0}$ so that no matter how large we take α, $\alpha V \not\supseteq AM$.

$$\Rightarrow \exists\, x_1 \in M \text{ so that } 1\cdot \overset{V}{\cancel{\alpha}} \not\ni Ax_1$$

$$\exists\, x_2 \in M \text{ so that } 2\cdot \cancel{\alpha} V \not\ni Ax_2$$

$$\vdots$$

$$\exists\, x_n \in M \text{ so that } n\cdot V \not\ni Ax_n$$

i.e. $\{\frac{1}{n}Ax_n\} \not\subset V. \Rightarrow \{A(\frac{x_n}{n})\} \not\subset V.$

But $\frac{x_n}{n} \to \vec{0}$ (*) and we've just shown that $A\left(\frac{x_n}{n}\right) \not\to \vec{0}$ in E_1. $\Rightarrow$ A is not sequentially continuous at $\vec{0}$ $\Rightarrow$ A is not continuous at $\vec{0}$.

(*) Here, I used the fact that $M \subseteq E$ is bounded if and only if given any sequence $\{x_n\} \subseteq M$ and any sequence $\{\varepsilon_n\}$ w/ $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ then $\varepsilon_n x_n \to \vec{0}$.

($\Leftarrow$) Again, assume not. I'll show that if A is not continuous then A is not bounded. Since E ~~is~~ satisfies the first axiom of countability $\exists$ countable local base in E at $\vec{0}$.

WLOG, $U_1 \supset U_2 \supset U_3 \supset \ldots$

Since A is linear, if A is not continuous at $x_0 \in E$ then A is not continuous at $\vec{0} \in E_1$ (this is because the topology in E and the topology in E_1 are generated by translating open sets ... since E and E_1 are topological vector spaces.) $\Rightarrow \exists$ a neighborhood $V \subset E_1$ of $\vec{0}$ and a sequence $\{x_n\} \subseteq E$ so that

$$x_n \in \tfrac{1}{n} U_n \quad \text{but} \quad Ax_n \notin V$$

The sequence $\{nx_n\}$ is bounded in E but the sequence $\{A(nx_n)\} = \{nAx_n\}$ is unbounded since $nAx_n \notin \alpha V \Rightarrow$ no matter how large you take α, $\exists\, n$ with $nAx_n \notin \alpha V$.

This proves that A is not a bounded operator, as desired. //

Recall, if E and E_1 are both normed vector spaces then A continuous $\Leftrightarrow$ A is bounded.

Also recall that $M \subset E$ bounded $\Leftrightarrow \exists R \ni x \in M \Rightarrow \|x\| \leq R$ and $AM \subset E_1$ bounded $\Leftrightarrow \exists R_1 \ni y \in AM \Rightarrow \|y\| \leq R_1$

by linearity of A, A is bounded $\Leftrightarrow \exists R_1$ so that

$$\|x\| \leq 1 \Rightarrow \|Ax\| \leq R_1$$

(Sound familiar? It should!)

So if $A: E \to E_1$ is bounded linear operator, we define its norm

$$\|A\|_{\mathcal{L}(E,E_1)} := \inf_{\|x\|_E \le 1} \|Ax\|_{E_1}$$

theorem:

$$\|A\|_{\mathcal{L}(E,E_1)} = \sup_{x \neq \vec{0}} \frac{\|Ax\|_{E_1}}{\|x\|_E}$$

$$\|Ax\|_{E_1} \le \|A\|_{\mathcal{L}(E,E_1)} \|x\|_E \qquad \forall x \in E$$

proof: Kolmogorov + Fomin

Note: I'm trying to be careful about putting subscripts on the norms so it's clearer which norm is meant. K+F doesn't do this.

Note: if A isn't defined on all of E, but on a subspace of E, then all of the above goes through
$\llcorner D_A$ just with the relative topology and the restriction $x \in D_A$

Assume $A: E \to E_1$ and $B: E \to E_1$ are linear & cts.
then $C: E \to E_1$ is linear & cts if
$C(x) := Ax + Bx \quad \forall x.$ $(C = A + B)$ if $D_A \subsetneq E$ or $D_B \subsetneq E$
then $D_C = D_A \cap D_B$

Similarly, if

$A: E \to E_1 \quad B: E_1 \to E_2$ are

continuous linear operators then

$C: E \to E_2$ defined by $Cx = ABx$

is a continuous linear operator.

And $A: E \to E_1$ continuous linear op $\Rightarrow$ $\alpha A: E \to E_1$ is a cont. linear op. for any $\alpha \in$ field K

If E, E_1, E_2 are normed vector spaces then

$$\| \alpha A \| = |\alpha| \, \|A\|.$$

$$\| A + B \| \leq \|A\| + \|B\|$$

$$\| AB \| \leq \|A\| \, \|B\|$$

Careful if $D_A \subsetneq E$ or $D_{AB} \subsetneq E$ (in defining those norms.)

Let $\mathcal{L}(E, E_1) = \{$linear operators that are continuous on E and $D_A = E\}$

If then $\mathcal{L}(E, E_1)$ is a vector space over our field K

if $\tilde{\mathcal{L}}(E, E) = \{$continuous linear operators with $D_A = E$ and A onto $E\}$

then $\tilde{\mathcal{L}}(E, E)$ is a ring over field K. (have + and $\cdot$; + is commutative, $\cdot$ doesn't have to be.)

theorem: Let E be a normed vector space and E_1 a Banach Space then $\mathcal{L}(E,E_1)$ is a Banach space

proof: We've already seen that $\mathcal{L}(E,E_1)$ is a vector space and that it has a norm. All we need to do is prove that it's a complete normed vector space.

Let $\{A_n\} \subseteq L(E,E_1)$ be a Cauchy sequence.

$\Rightarrow$ given $\varepsilon > 0$ $\exists N_\varepsilon$ so that $m,n \geq N$

$\Rightarrow$ $\|A_n - A_m\|_{\mathcal{L}(E,E_1)} < \varepsilon$.

Fix $x \in E$. Since $\|A_n x - A_m x\|_{E_1} \leq \|A_n - A_m\|_{\mathcal{L}(E,E_1)} \|x\|_E$

we see that $\{A_n\}$ cauchy in $\mathcal{L}(E,E_1)$ implies $\{A_n x\}$ Cauchy in E_1. Since E_1 is complete, $\exists$ $y \in E_1$ with $A_n x \to y$. Define $Ax = y$.

In this way, for each $x \in E$, we define $Ax \in E_1$. A is clearly linear. We just want to show it's a bounded linear operator and that $A_n \to A$ in $\mathcal{L}(E,E_1)$. (Recall! we defined A pointwise this doesn't automatically imply that $\|A_n - A\|_{\mathcal{L}(E,E_1)} \to 0$ as $n \to \infty$!)

First, show $A_n \to A$. Fix $\varepsilon > 0$, Take $N_\varepsilon \ni k, n \geq N_\varepsilon \Rightarrow \|A_m - A_n\| < \varepsilon$

$$\|A_n - A\|_{\mathcal{L}(E,E_1)} = \sup_{\|x\|_E = 1} \|A_n x - Ax\|_{E_1} \quad \text{(if supremum is finite.)}$$

$$= \sup_{\|x\|_E = 1} \lim_{k\to\infty} \|A_n x - A_k x\|_{E_1}$$

$$\leq \sup_{\|x\|_E = 1} \lim_{k\to\infty} \|A_n - A_k\|_{\mathcal{L}(E,E_1)} \|x\|_E$$

$$\leq \sup_{k \geq N_\varepsilon} \|A_n - A_k\|_{\mathcal{L}(E,E_1)} < \varepsilon.$$

$\Rightarrow \|A_n - A\|_{\mathcal{L}(E,E_1)} < \varepsilon$ if $n \geq N_\varepsilon$.

Note: if you fix n, this proves $A \in \mathcal{L}(E,E_1)$ since

$$\|A\|_{\mathcal{L}(E,E_1)} := \sup_{\|x\| \leq 1} \|Ax\|_{E_1} = \sup_{\|x\| \leq 1} \|(A - A_n)x + A_n x\|$$

$$\leq \sup_{\|x\| \leq 1} \|(A - A_n)x\|_{E_1} + \sup_{\|x\| \leq 1} \|A_n x\|$$

$$< C_1 < \infty$$

by what we did before. (Fix $n \geq N_\varepsilon$, get RHS $\leq \varepsilon + \|A_n\|_{\mathcal{L}(E,E_1)} < \infty$.)

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Notice embedded in that proof was the following:

If we have a continuous linear operator defined on a convergent sequence $\{x_n\}$ then we can *extend* A to the limit point x, by $Ax := \lim_{n\to\infty} Ax_n$.

This is a glaringly obvious step but extremely important.

Q: How do you define the Fourier transform for all $\phi \in L^2$? (I said that $\mathcal{F}: L^2 \to L^2$ is a bounded linear functional.)

A: Find a dense subspace in L^2 where the functions in the subspace L_0 are nice enough that you can define $\mathcal{F}$. Then prove that $\mathcal{F}$ is a bounded linear functional $L_0 \to L^2$. Finally, extend $\mathcal{F}$ to all of L^2 by the density of L_0. The extended $\mathcal{F}$ is a bounded linear functional from $L^2 \to L^2$

Q: So what's a good subspace that's dense in L^2?

A: Wait until measure theory...

Given $A: E \to E_1$, we can define its null space

$$N_A = \{x \in E \mid Ax = \vec{0}\}$$

$\vec{0}$ is closed subset of E_1, A is continuous

$$\Rightarrow N_A = A^{-1}\{0\} \text{ is a closed subset of } E.$$

Clearly N_A is a subspace, so N_A is a closed <u>subspace</u> of E.

<u>Theorem</u>: Let E and E_1 be normed vector spaces and $A: E \to E_1$ a bounded linear operator

Then (1) N_A, the nullspace of A, is a closed subspace of E

(2) A, when regarded as a map

$$A_0 : E/N_A \to E_1$$

is a 1:1 linear mapping, is bounded, and $\|A_0\|_{\mathcal{L}(E/N_A, E_1)} = \|A\|_{\mathcal{L}(E, E_1)}$. The range of A is the same as the range of A_0.

proof.

1) $N_A = A^{-1}\{0\}$ is closed since $\{0\}$ is closed and A is continuous

2) First, recall E/N_A

$x \equiv y$ if $x - y \in N_A$

i.e. $x \equiv y$ if $Ax = Ay$ ⊛

Let $\{x\}$ be the equivalence class of x.

by ⊛, $A\{x\} = Ax$ is a well-defined function from E/N_A to E_1. A linear $\Rightarrow A$

E/N_A is a normed vector space with norm

$$\|\{x\}\|_{E/N_A} := \inf_{y \in N_A} \|x+y\|_E$$

(Since N_A is a closed subspace, this is a norm.)

So all we need to do is show that A_0 is a bounded linear functional from E/N_A to E_1

and that

$$\|A_0\|_{\mathcal{L}(E/N_A, E_1)} = \|A\|_{\mathcal{L}(E, E_1)}$$

$$\|A\|_{\mathcal{L}(E,E_1)} = \sup_{x \neq 0} \frac{\|Ax\|_{E_1}}{\|x\|_E}$$

$$= \sup_{x \neq 0} \sup_{y \in N_A} \frac{\|A(x+y)\|_{E_1}}{\|x+y\|_E}$$

$$= \sup_{x \neq 0} \sup_{y \in N_A} \frac{\|Ax\|_{E_1}}{\|x+y\|_E} \qquad \text{since } Ay = 0$$

$$= \sup_{x \neq 0} \frac{\|Ax\|_{E_1}}{\inf_{y \in N_A} \|x+y\|_E}$$

$$= \sup_{x \neq 0} \frac{\|Ax\|_{E_1}}{\|\{x\}\|_{E/N_A}} = \|A_0\|_{\mathcal{L}(E/N_A, E_1)}$$

and we're done! //

Note: in the above we proved A_0 is a bounded linear operator by proving

$$\sup_{x \neq 0} \frac{\|Ax\|_{E_1}}{\|\{x\}\|_{E/N_A}}$$ is finite. It just so happens

to equal $\|A\|_{\mathcal{L}(E,E_1)}$ as desired. :)