Last time, I introduced three test function spaces

$$C^\infty(\mathbb{R}) \supseteq \mathcal{S}(\mathbb{R}) \supseteq C_0^\infty(\mathbb{R})$$

and I gave examples of linear functions that went from

X = a function space where its members satisfy a certain # of conditions

to $\mathbb{R}$ or $\mathbb{C}$. I did not discuss whether these linear functionals were <u>continuous</u> since that requires a <u>topology</u> on the test function space.

<u>recall</u>:

$\phi \in C^\infty(\mathbb{R})$ if ϕ is continuous and $\frac{\partial^\alpha \phi}{\partial x^\alpha}$ exists and is continuous for all $\alpha \geq 1$.

<u>Note</u>: It's possible for $\frac{\partial^\alpha \phi}{\partial x^\alpha}$ to exist for all $\alpha \geq 1$ but for ϕ not to be continuous (HW!)

Now we define a topology on $C^\infty(\mathbb{R})$ via <u>seminorms</u>

Let L be a complex vector space

$$p : L \to \mathbb{R}$$

is a <u>seminorm</u> if

(1) $p(x) \geq 0 \quad \forall x \in L$

(2) $p(\alpha x) = |\alpha| p(x) \quad \forall x \in L \quad \forall \alpha \in (\mathbb{R} \text{ or } \mathbb{C})$

(3) $p(x+y) \leq p(x) + p(y)$

<u>Note:</u> the only difference between a seminorm and a norm is that a norm satisfies a nondegeneracy condition:

(4) $p(x) = 0 \iff x = 0$

Let $L = C^\infty(\mathbb{R})$ Fix $k \geq 0, m \geq 0 \quad k, m \in \mathbb{N}$

$$P_{m,k}(\phi) = \max_{0 \leq j \leq k} \sup_{-m \leq x \leq m} |\phi^{(j)}(x)|$$

recall <u>sub-base</u>:

let $\mathcal{B}_0$ be a collection of subsets of L so that $\bigcup_{B \in \mathcal{B}_0} B = L$. Let $\mathcal{B} = \{$all finite intersections of elts of $\mathcal{B}_0\}$

then $\mathcal{B}$ is a base for a topology.

The <u>topology</u> for $C^\infty(\mathbb{R})$ is given by the sub-base

$$B_{m,k,\varepsilon}(\phi) := \{ \psi \mid P_{m,k}(\phi - \psi) < \varepsilon \}$$

<u>theorem</u>: $\{\phi_n\} \subseteq C^\infty(\mathbb{R})$, $\phi \in C^\infty(\mathbb{R})$.

Then $\phi_n \to \phi$ (in the topology just given) if and only if given $\varepsilon > 0$, $K, k \in \mathbb{N}$ $\exists$ $N_{\varepsilon,K,k}$ so that $n \geq N_{\varepsilon,K,k}$ then

$$P_{K,k}(\phi_n - \phi) < \varepsilon$$

<u>proof</u>: follows directly from definition of subbase and defn of $P_{M,k}$.

i.e. $\phi_n \to \phi$ if on any closed ball $\{x \mid |x| \leq K\}$ all partial derivatives of $\phi_n - \phi$ (up to a specified order) can be made as small as desired.

<u>Note</u>: $(C^\infty(\mathbb{R}), \tau)$ can be <u>metrized</u>

$$d(\phi,\psi) = \sum_{m=1}^{\infty} \frac{P_{m,m}(\phi-\psi)}{1+P_{m,m}(\phi-\psi)} \frac{1}{2^m}$$

but cannot be normed.

Since $(C^\infty(\mathbb{R}),\tau)$ can be metrized, if $\phi_n \to \phi \Rightarrow T(\phi_n) \to T(\phi)$ then this is sufficient to show continuity of T

<u>claim</u>. $\delta_0(\phi) := \phi(0)$ is a continuous linear functional on $(C^\infty(\mathbb{R}), \tau)$.

<u>proof 1</u>. Since $(C^\infty(\mathbb{R}), \tau)$ is a metric space, it suffices to show that if $\phi_n \to \phi$ then $\delta_0(\phi_n) \to \delta_0(\phi)$.

Fix $\varepsilon > 0$. By theorem, $\exists$ K and k ~~so that~~ and $\exists N$ so that $n \geq N \Rightarrow$

$$P_{K,k}(\phi_n - \phi) < \varepsilon.$$

i.e. $$\max_{0 \leq i \leq k} \sup_{|x| \leq K} \left| \frac{\partial^i}{\partial x^i}(\phi_n - \phi) \right| < \varepsilon.$$

specifically, take $i=0$. Then

$$\sup_{|x| \leq K} |\phi_n(x) - \phi(x)| < \varepsilon.$$

specifically, take $x=0$. Then

$$|\phi_n(0) - \phi(0)| < \varepsilon$$

i.e. $|\delta_0(\phi_n) - \delta_0(\phi)| < \varepsilon \Rightarrow \delta_0$ is cts : $(C^\infty, \tau) \to (\mathbb{R}, |\cdot|)$.

<u>proof 2</u>: Let $W \subset \mathbb{R}$ be an open nbd of $\delta_0(\phi) = \phi(0)$.

Want to show $\delta_0^{-1}(W)$ is open in $C^\infty(\mathbb{R})$.

i.e. if $\psi \in \delta_0^{-1}(W)$ then $\exists\, U \in \tau$ so that $\psi \in U \subseteq \delta_0^{-1}(W)$. i.e. $\delta_0(U) \subseteq W$ and $\psi \in U$.

Let $\psi \in \delta_0^{-1}(W)$.

Since W is open in $\mathbb{R}$, $\exists\ \varepsilon > 0$ so that

$S(\psi(0), \varepsilon) \subseteq W$ (the open ball around $\phi(0)$ is contained in W.)

Let $U = B_{1,0,\varepsilon}(\psi) = \{\tilde{\phi} \mid P_{1,0}(\tilde{\phi} - \psi) < \varepsilon\}$

$$= \{\tilde{\phi} \mid \max_{0 \le i \le 0} \sup_{|x| \le 1} \left|\frac{\partial^i}{\partial x^i}(\tilde{\phi} - \psi)\right| < \varepsilon\}$$

$$= \{\tilde{\phi} \mid \sup_{|x| \le 1} |\tilde{\phi}(x) - \psi(x)| < \varepsilon\}.$$

<u>Note</u>: $\psi \in U$ ✓

<u>Note</u>: $U \in \mathcal{T}$ ✓

<u>Note</u>: if $\tilde{\phi} \in U$ then $|\tilde{\phi}(0) - \psi(0)| < \varepsilon$

$$\Rightarrow |\delta_0(\tilde{\phi}) - \delta_0(\psi)| < \varepsilon$$

$$\Rightarrow \delta_0(\tilde{\phi}) \in S(\psi(0), \varepsilon) \subseteq W$$

$$\Rightarrow \delta_0(U) \subseteq W$$ and done!

Okay, so our friend the δ-function is continuous on $(C^\infty(\mathbb{R}), \mathcal{T})$. No shock there.

The Cauchy Principal Value $PV\frac{1}{x}$ functional, however, isn't even automatically defined for all $\phi \in C^\infty(\mathbb{R})$ since $PV\frac{1}{x}(\phi) := \lim_{\varepsilon \downarrow 0} \int_{|x| > \varepsilon} \frac{\phi(x)}{x}dx$ and integral might not conv.

What about a topology on $C_0^\infty(\mathbb{R})$?
It's more complicated... not only does it have to keep track of all derivatives but it has to notice the support as well

---

Let $\vec{a} = \{a_0, a_1, a_2, \dots\}$ be a nondecreasing sequence of positive real numbers. Let $\vec{k} = \{k_0, k_1, k_2 \dots\}$ be a nondecreasing sequence of nonnegative integers define

$$P_{\vec{a},\vec{k}}(\phi) = \sup_{j \geq 0} \sup_{|\alpha| \leq k_j} \sup_{|x| \geq j} a_j \left| \frac{\partial^\alpha}{\partial x^\alpha} \phi(x) \right|$$

Then $B_{\vec{a},\vec{k},\varepsilon}(\phi) = \{ \psi \mid P_{\vec{a},\vec{k}}(\phi - \psi) < \varepsilon \}$

is a sub-base for the topology on $C_0^\infty(\mathbb{R})$

OUCH!! That's a hard seminorm to understand.

<u>Note</u>: if ϕ is not compactly supported then for each j $\exists |x_j| > j$ so that $\phi(x_j) \neq 0$.
This allows you to choose a sequence $\vec{a}$ so that

$$a_j \phi(x_j) > \text{as big a number as you want}$$

$\Rightarrow P_{\vec{a},\vec{k}}(\phi - \vec{0}) \not< \varepsilon$. Cannot make ϕ "close" to 0.

<u>theorem</u>: Let $\{\phi_n\} \subseteq C_0^\infty(\mathbb{R})$

1) assume $\exists$ R so that

$$|x| > R \Rightarrow \phi_j(x) = 0 \qquad (R \text{ does not depend on } j\,!)$$

2) for each k,

$$\max_{0 \le i \le k} \; \sup_{|x| \le R} \left| \frac{\partial^i}{\partial x^i} \phi_n(x) \right| \to 0 \quad \text{as } n \to \infty$$

<u>Then</u> $\phi_n \to 0$ in $(C_0^\infty(\mathbb{R}), \tau)$.

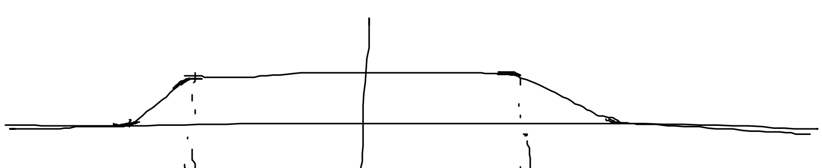
So the difference between $\phi_n \to 0$ in $(C^\infty(\mathbb{R}), \tau)$

and $\phi_n \to 0$ in $(C_0^\infty(\mathbb{R}), \tau)$

is 1) --- we require that all the ϕ_n have their support in the same bounded set.

e.g. define

$$\phi_j(x) = \begin{cases} \frac{1}{2j} & \text{if } |x| \le j \\ \in [0, \frac{1}{2j}] & \text{if } |x| \in [j, j+1] \\ 0 & \text{if } |x| \ge j+1 \end{cases}$$



Then $\phi_j \not\to 0$ in $C_0^\infty(\mathbb{R})$.

Okay, so $\phi_j \not\to 0$. Is this so fatal?

Consider the linear functional

$$T(\phi) := \lim_{R\to\infty} \int_{-R}^{R} \phi(x)\,dx$$

This should be a reasonable function,

especially for nice functions like my ϕ_j's.

$$\text{But } T(\phi_j) > \int_{-j}^{j} \phi_j(x)\,dx = \int_{-j}^{j} \frac{1}{2j}\,dx = 1$$

$$\text{And } T(0) = \lim_{R\to\infty} \int_{-R}^{R} 0\cdot dx = 0$$

So it's a good thing that $\phi_j \not\to 0$ since if my $\phi_j \to 0$ then this (nice and desirable) linear functional T would not be continuous.

Similarly $\phi_n(x) = \begin{cases} 1 & \text{if } x \in [n, n+1] \\ 0 & \text{if } x < n - 1/2 \text{ or } x > n + 3/2 \\ \text{smooth inbetween} & \end{cases}$

doesn't have $\phi_n \to 0$ in $C^\infty(\mathbb{R})$.

<u>Claim</u>: $PV\frac{1}{x} : C_0^\infty(\mathbb{R}) \to \mathbb{C}$ is sequentially continuous

<u>proof</u>: Since $PV\frac{1}{x}$ is linear, it suffices to prove continuity at 0. Let $\phi \in C_0^\infty(\mathbb{R})$. From last time,

$$PV\frac{1}{x}\,\phi = \int_{-\infty}^{-1} \frac{\phi(x)}{x}\,dx + \int_{-1}^{1} \frac{\phi(x)-\phi(0)}{x}\,dx + \int_{1}^{\infty} \frac{\phi(x)}{x}\,dx$$

$$\Rightarrow \left| PV\frac{1}{x}(\phi) \right| \le \int_{-R}^{-1} \left|\frac{\phi(x)}{x}\right| dx + \int_{-1}^{1} \left|\frac{\phi(x)-\phi(0)}{x}\right| dx + \int_{1}^{R} \left|\frac{\phi(x)}{x}\right| dx$$

note: $\displaystyle\int_{-R}^{-1} \left|\frac{\phi(x)}{x}\right| dx \le \int_{-R}^{-1} |\phi(x)|\,dx \le \boxed{(R-1)\,P_{R,0}(\phi)}$!! careful !! this is $P_{\vec{a},\vec{k}}$ where $\vec{k}$ = all 1's and $\vec{a}$ = ???

similarly for the $\int_1^R$ integral.

$$\int_{-1}^{1} \left|\frac{\phi(x)-\phi(0)}{x}\right| dx = \int_{-1}^{1} |\phi'(c_x)|\,dx \qquad \text{(mean value theorem)}$$

$$\le \int_{-1}^{1} P_{1,1}(\phi)\,dx = \boxed{2\cdot P_{1,1}(\phi)}$$!! careful !! similar caveat...

$$\Rightarrow \left| PV\frac{1}{x}(\phi) \right| \le 2(R-1)\,P_{R,0}(\phi) + 2\,P_{1,1}(\phi)$$

Okay, now we've found how to relate $PV\frac{1}{x}\phi$ to some seminorms.

But they're the <u>wrong</u> seminorms!

recall our theorem though...

$$\phi_n \to 0 \quad \text{in } C_0^\infty(\mathbb{R})$$

if ① $\exists R$ so that $|x| > R \Rightarrow |\phi_n(x)| = 0$

② $P_{\tilde{R},k}(\phi_n) \to 0$ as $n \to \infty$ for $\forall \tilde{R}, \forall k$

And now we see that our bounds allow us to prove sequential continuity of $PV\frac{1}{x}$ easily!

assume $\phi_n \to 0$. Then $\exists R$ so that $|\phi_n(x)| = 0 \quad \forall n \; \forall |x| > R$.

We want to show $PV\frac{1}{x}(\phi_n) \to 0$. i.e. prove that given $\varepsilon > 0 \; \exists N$ so that $n \geq N$ $\Rightarrow \left| PV\frac{1}{x}(\phi_n) \right| < \varepsilon$. We know

$$\left| PV\frac{1}{x}(\phi_n) \right| \leq 2(R-1) P_{R,0}(\phi_n) + 2 P_{1,1}(\phi_n)$$

and since $\phi_n \to 0$ in $C_0^\infty(\mathbb{R})$ $P_{R,0}(\phi_n)$ & $P_{1,1}(\phi_n)$ both $\to 0$. So ~~then~~ take N so that

$$n \geq N \Rightarrow P_{R,0}(\phi_n) < \frac{\varepsilon}{4(R-1)} \text{ and } P_{1,1}(\phi_n) < \frac{\varepsilon}{4}$$

and we're done! //

Again, we use seminorms to help us define the topology on $\mathcal{S}(\mathbb{R})$.

$$q_{m,k}(\phi) = \max_{|\alpha| \le k} \sup_{x \in \mathbb{R}} |x|^m \left| \frac{\partial^\alpha}{\partial x^\alpha} \phi(x) \right|$$

this is a much friendlier seminorm than the $C_0^\infty(\mathbb{R})$ seminorms.

Theorem: $\phi_j \to \phi$ in $\mathcal{S}$ if $q_{m,k}(\phi_j - \phi) \to 0$ as $j \to \infty$ $\forall m, \forall k$.

Let's move on. Note in Kolmogorov + Fomin, they use the notation

$$K := C_0^\infty(\mathbb{R})$$

and say $\phi \in C_0^\infty(\mathbb{R}) \iff$ "ϕ is finite" (ouch!)

Also, note when talking about generalized function we don't worry about continuity with respect to $\mathcal{T}$, we refer to sequential continuity

defn: Every continuous linear functional $T(\phi)$ on K is called "a generalized function on $(-\infty,\infty)$" where the continuity of $T(\phi)$ means that $\phi_n \to \phi$ in $K \Rightarrow T(\phi_n) \to T(\phi)$.

OOOF! We don't need to worry about the topology!

If we want to think about generalized functions converging to each other, we use weak* convergence

$$T_n \xrightarrow[w*]{} T \iff T_n(\phi) \to T(\phi) \quad \forall \phi \in K.$$

Notation: will often denote T by "f" as if a representation of the form

$$T(\phi) = (f,\phi) = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx$$

actually made sense

This is helpful though in that we can define the derivative of a generalized function by its action $\frac{dT}{dx}(\phi) = -T\left(\frac{d\phi}{dx}\right)$

$$\frac{d^2T}{dx^2}(\phi) = T\left(\frac{d^2\phi}{dx^2}\right)$$

$$\frac{d^kT}{dx^k}(\phi) = (-1)^k T\left(\frac{d^k\phi}{dx^k}\right) \qquad \forall k \in \mathbb{N}.$$

Where did that come from?
Well, it certainly makes sense on a <u>formal level</u>:

$$\frac{dT}{dx}(\phi) = \left(\frac{df}{dx}, \phi\right) = \int_{-\infty}^{\infty} \frac{df}{dx}(x)\,\phi(x)\,dx$$

$$= f(x)\phi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\frac{d\phi}{dx}(x)\,dx$$

$$= -\int_{-\infty}^{\infty} f(x)\frac{d\phi}{dx}(x)\,dx \qquad \text{since } \phi' \text{ has compact support}$$

$$= -\left(f, \frac{d\phi}{dx}\right)$$

$$= T\left(\frac{d\phi}{dx}\right)$$

and $\frac{dT}{dx}$ is certainly linear.

<u>Q:</u> is $\frac{dT}{dx}$ sequentially continuous? i.e. if $\phi_n \to \phi$ does this imply $\frac{dT}{dx}(\phi_n) \to \frac{dT}{dx}(\phi)$? i.e. does this imply $T\left(\frac{d\phi_n}{dx}\right) \to T\left(\frac{d\phi}{dx}\right)$? Yes! Since $\phi_n \to \phi$ in $C_0^\infty(\mathbb{R})$ means that $\frac{d\phi_n}{dx} \to \frac{d\phi}{dx}$ in $C_0^\infty(\mathbb{R})$ and then we're done by the sequential continuity of T!

This is very powerful!

For example, say you've been given $f(x)$ continuous on $\mathbb{R}$ and you want to find $u(x)$ on $\mathbb{R}$ so that

$$\frac{\partial^2 u}{\partial x^2}(x) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

i.e. u solves the PDE $\frac{\partial^2 u}{\partial x^2} = f$.

If we can find a u that has two derivatives and the second derivative is continuous and equals f then we're thrilled. We call u "a classical solution".

If, on the other hand, we can find a generalized function T so that for all $\phi \in C_0^\infty(\mathbb{R})$ we have

$$\left(\frac{d^2T}{dx^2}(\phi) = \right) \quad T\left(\frac{d^2\phi}{dx^2}\right) = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$$

then we've found a "weak solution". Note: we didn't even need $f \in C(\mathbb{R})$. We can define a weak solution of $\frac{\partial^2 u}{\partial x^2} = f$ where f is a generalized function

if we find T so that

$$T\left(\frac{d^2\phi}{dx^2}\right) = f(\phi) \quad \forall \phi \in C_0^\infty(\mathbb{R}).$$