HW #9.

Problem 1:

a). Let $f \in L^p$ and $g \in L^q$.

$f_{\sigma\mu}(x) := \mu f(\sigma x)$ $\quad g_{\sigma\mu} := \mu g(\sigma x)$ $\quad \mu, \sigma > 0$.

$$\|f_{\sigma\mu} g_{\sigma\mu}\|_{L^1} = \int |\mu f(\sigma x) \mu g(\sigma x)| dx = \mu^2 \int |f(\sigma x) g(\sigma x)| dx$$

$$= \mu^2 \sigma^{-n} \int |f(y) g(y)| dy$$

$$= \mu^2 \sigma^{-n} \|fg\|_{L^1}$$

$$\|f_{\sigma\mu}\|_{L^p} = \sqrt[p]{\int |\mu f(\sigma x)|^p dx} = \mu \sqrt[p]{\int |f(\sigma x)|^p dx}$$

$$= \mu \sigma^{-n/p} \|f\|_{L^p}$$

$$\|g_{\sigma\mu}\|_{L^q} = \mu \sigma^{-n/q} \|g\|_{L^q}.$$

Fix f and $g \in L^p, L^q$ respectively.

then $f_{\sigma\mu} \in L^p$, $g_{\sigma\mu} \in L^q$ $\forall \mu, \sigma > 0$

Can $\|fg\|_{L^1} \leq C \|f\|_{L^p} \|g\|_{L^q}$ hold $\forall f \in L^p, g \in L^q$?
If yes, then it must hold for $f_{\sigma\mu}$ and $g_{\sigma\mu}$. If it doesn't hold for $f_{\sigma\mu}$ and $g_{\sigma\mu}$ then it certainly won't hold for all $f, g \in L^p, L^q$.

If the inequality holds for all $f_{\sigma,\mu}$, $g_{\sigma,\mu}$ then

$$\mu^2 \sigma^{-n} \|fg\|_{L^1} \le C \mu^2 \sigma^{-n/p} \sigma^{-n/q} \|f\|_{L^p} \|g\|_{L^q}.$$

At this point, f and g are fixed and $C < \infty$. We see that μ^2 cancels out of the inequality.

If $\sigma^{-n} \neq \sigma^{-n/p}\sigma^{-n/q}$ then we have

$$\|fg\|_{L^1} \le \underbrace{C \|f\|_{L^p} \|g\|_{L^q}} \cdot \sigma^{n - \frac{n}{p} - \frac{n}{q}}$$

a fixed # a fixed #.

if $n - \frac{n}{p} - \frac{n}{q} < 0$ then by taking $\sigma \to \infty$ the RHS will get as small as we want and eventually the inequality will fail.

if $n - \frac{n}{p} - \frac{n}{q} > 0$ then by taking $\sigma \to 0$ the RHS will get as small as we want and eventually the inequality will fail.

conclusion: if $\frac{1}{p} + \frac{1}{q} \neq 1$ then it's impossible to find $C < \infty$ so that

$$\|fg\|_{L^1} \le C \|f\|_{L^p} \|g\|_{L^q} \quad \forall f \in L^p \ \forall g \in L^q.$$

b) Assume $u \in L^{p^*}$ and $\nabla u \in L^p$

then $u_{\sigma\mu}(x) = \mu u(\sigma x) \in L^{p^*}$

and $\nabla_x u_{\sigma\mu} \in L^p$.

from part a) we know $\|u_{\sigma\mu}\|_{L^{p^*}} = \mu\sigma^{-n/p^*}\|u\|_{L^p}$

$$\nabla_x u_{\sigma\mu}(x) = \nabla_x \mu u(\sigma x)$$

$$= \mu\sigma \nabla_y u(\sigma x) \quad \text{where } y = \sigma x.$$

$$\Rightarrow \|\nabla u_{\sigma\mu}\|_{L^p} = \mu\sigma\sigma^{-n/p}\|\nabla u\|_{L^p}$$

Fix $u \in L^{p^*}$ with $\nabla u \in L^p$. if $\exists\, C < \infty$ so that

$$\|u\|_{L^{p^*}} \le C\|\nabla u\|_{L^p}$$

for all $u \in X = \{u \in L^{p^*} \mid \nabla u \in L^p\}$ then it will certainly be true for $u_{\sigma\mu}$. And if it fails for $u_{\sigma\mu}$ then it won't be true for arbitrary $u \in X$.

$$\mu\sigma^{-n/p^*}\|u\|_{L^{p^*}} \le C\mu\sigma\sigma^{-n/p}\|\nabla u\|_{L^p}$$

$$\|u\|_{L^{p^*}} \le C\sigma^{1-\frac{n}{p}+\frac{n}{p^*}}\|\nabla u\|_{L^p}$$

by same logic as before, need $p^* = \dfrac{np}{n-p}$ if the inequality will have any chance of holding.

c) Let $X = \{ u \in C^\alpha \mid \nabla u \in L^p \}$.

fix $u \in X$. then $u_{\sigma\mu} \in X \quad \forall \sigma, \mu > 0$.

$$[u_{\sigma\mu}]_\alpha = \sup_{x \neq y} \frac{|u_{\sigma\mu}(x) - u_{\sigma\mu}(y)|}{|x-y|^\alpha}$$

$$= \mu \sup_{x \neq y} \frac{|u(\sigma x) - u(\sigma y)|}{|x-y|^\alpha}$$

$$= \mu \sigma^\alpha \sup_{x \neq y} \frac{|u(\sigma x) - u(\sigma y)|}{|\sigma x - \sigma y|^\alpha} = \mu \sigma^\alpha [u]_\alpha$$

from b), $\|\nabla u_{\sigma\mu}\|_{L^p} = \mu \sigma \sigma^{-n/p} \|\nabla u\|_{L^p}$.

if $\quad [u]_\alpha \leq C \|\nabla u\|_{L^p} \quad \forall u \in X$ for some $C < \infty$

then $\quad \mu \sigma^\alpha [u]_\alpha \leq C \mu \sigma \sigma^{-n/p} \|\nabla u\|_{L^p}$

$\Rightarrow$ if $\alpha \neq 1 - \frac{n}{p}$ then the inequality cannot possibly hold.

d) We have

$$\mathcal{F} f(\xi) := \frac{1}{\sqrt{2\pi}} \int f(x) e^{ix \cdot \xi} \, dx$$

$\mathcal{F} : L^p \to L^q$ will be continuous if $\exists C < \infty$ so that

$$\|\mathcal{F} f\|_{L^q} \leq C \|f\|_{L^p} \qquad \forall f \in L^p$$

fix $f \in L^p$, then $f_{\sigma\mu} = \mu f(\sigma x) \in L^p$ $\forall \mu > 0, \sigma > 0$

and $\|f_{\mu\sigma}\|_{L^p} = \mu \sigma^{-n/p} \|f\|_{L^p}$

$$\mathcal{F} f_{\mu\sigma}(\xi) = \frac{1}{\sqrt{2\pi}} \int f_{\mu\sigma}(x) e^{ix\cdot\xi} dx$$

$$= \frac{\mu}{\sqrt{2\pi}} \int f(\sigma x) e^{ix\cdot\xi} dx = \frac{\mu}{\sqrt{2\pi}} \int f(\sigma x) e^{i\sigma x \cdot \frac{\xi}{\sigma}} dx$$

$$= \frac{\mu}{\sqrt{2\pi}} \sigma^{-n} \int f(y) e^{iy\cdot\frac{\xi}{\sigma}} dy$$

$$= \mu \sigma^{-n} \mathcal{F} f\left(\frac{\xi}{\sigma}\right).$$

$$\Rightarrow \|\mathcal{F} f_{\mu\sigma}\|_{L^q} = \mu \sigma^{-n} \sigma^{n/q} \|\mathcal{F} f\|_{L^q}$$

So we want

$$\mu \sigma^{-n} \sigma^{n/q} \|\mathcal{F} f\|_{L^q} \le C \mu \sigma^{-n/p} \|f\|_{L^p} \quad \forall \mu, \sigma$$

i.e. $-n + \frac{n}{q} = -\frac{n}{p}$ needs to hold.

i.e. $\frac{1}{p} + \frac{1}{q} = 1$ needs to hold. ✓

Problem 2: Let X be a locally compact Hausdorff topological space. We want to prove the Baire Category theorem for X.

defn. X is a locally compact Hausdorff topological vector space if X is Hausdorff and if for each $x \in X$, $\exists\ K \subseteq X$ where K is compact and $x \in K^\circ$. (K is a "compact neighborhood" of x.)

lemma: Let X be a LCH space and $x \in U \in \mathcal{T}$. then $\exists\ K$ compact so that $x \in K^\circ$ and $K \subseteq U$.

proof: Since X is LCH, $\exists K$ compact with $x \in K^\circ$. Without loss of generality we can assume $[U]$ is compact. (If not, then replace U with $U \cap K^\circ$. It continues to be an open set containing x and $[U \cap K^\circ] \subseteq [K^\circ] = K$ is compact $\Rightarrow [U \cap K^\circ]$ is a closed subset of a compact set, hence is compact.)

$\partial U = [U] - U$ is a compact set (because it's a closed subset of a compact set.)

Since ∂U is compact and $x \notin \partial U$ and X is Hausdorff, $\exists$ open sets V and W w/ $V \cap W = \emptyset$

and $x \in V$ and $\partial U \subseteq W$.

By replacing V with $V \cap U$, we can further assume that $V \subseteq U$.

I claim $[V] \subseteq U$. It will then follow that we have our desired set V so that V is open $x \in V \subseteq [V] \subseteq U$ where $[V]$ is compact.

assume $[V] \nsubseteq U$. Since $V \subset U$ we know $[V] \subseteq [U]$.

$\therefore$ if $[V] \nsubseteq U$ then $\exists \{v_n\} \subseteq V$ with $v_n \to v$ where $v \notin U$. we know $v \in [U] \Rightarrow v \in \partial U$.

$\therefore$ $v_n \to v \in \partial U$. & Since W is an open set containing ∂U, $\exists N$ so that $n \geq N \Rightarrow v_n \in W$. $\times$ since $V \cap W = \phi$.

Therefore $v \in U$ and $[V] \subseteq U$ and done. //

Now we're ready to prove our desired theorem

<u>theorem</u>: if X is LCH topological vector space and $A \subseteq X$ is a residual set then A is dense in X.

<u>proof</u>: A is residual $\Rightarrow$ $A = \bigcap_{1}^{\infty} U_n$ where $(U_n)^{\circ}$ is dense in X. To show A is dense, we want to show that if V is a nonempty open set then $A \cap V \neq \phi$.

Since $A = \bigcap_1^\infty U_n \supseteq \bigcap_1^\infty (U_n)^\circ$ it suffices to show that $V \cap \left(\bigcap_1^\infty (U_n)^\circ\right)$ is nonempty.

Because $(U_1)^\circ$ is dense, $V \cap (U_1)^\circ \neq \phi$. $\Rightarrow \exists x_1 \in V \cap (U_1)^\circ$ and $\exists$ an open set V_1 so that $x \in V_1 \subset (V \cap (U_1)^\circ)$ where $[V_1]$ is compact and $[V_1] \subseteq (V \cap (U_1)^\circ)$ (By my lemma)

Because $(U_2)^\circ$ is dense $V_1 \cap (U_2)^\circ \neq \phi$. $\Rightarrow \exists x_2 \in V_1 \cap (U_2)^\circ$. and $\exists$ an open set V_2 so that

$$x_2 \in V_2 \subset (V_1 \cap (U_2)^\circ) \subseteq V \cap (U_1)^\circ \cap (U_2)^\circ$$

and $[V_2]$ is compact. and $[V_2] \subseteq (V \cap (U_1)^\circ \cap (U_2)^\circ)$.

Proceeding inductively, we construct a nested sequence of compact sets so that

$$V_n \subseteq V \cap \left(\bigcap_1^n (U_k)^\circ\right).$$

and $[V_n]$ is compact, $[V_n] \subseteq V \cap \left(\bigcap_1^n (U_k)^\circ\right)$

I claim that $\bigcap_1^n [V_k] \neq \phi$ for each n since they are all a subset of $[V_1]$ which is compact, it then follows that $\bigcap_1^\infty [V_k] \neq \phi$.

i.e. $\exists x_\infty \in \bigcap_1^\infty [V_n] \subseteq V \cap \left(\bigcap_1^\infty (U_n)^\circ\right)$ and done!

But the finite intersection property is automatic by our construction:

$$V_n \subset V_{n-1} \subset \cdots V_3 \subset V_2 \subset V_1$$

Therefore A is dense as desired. //

Note X is not a meagre set.

Why? assume not. Assume X is meagre. then X^c is residual. i.e. ϕ is residual. $\Rightarrow \phi$ is dense in X. This is impossible (unless X itself is empty.). Therefore X is not meagre.

Problem 3: Assume $A \in \mathcal{L}(E,F)$ is onto F, let $B \in \mathcal{L}(E,F)$. Show that

$A+B$ is onto if B is "small".

By the open mapping theorem, $\exists\ \delta > 0$ so that $B_\delta(0) \subseteq AB_1(0)$.

i.e if $\|z\| \le \delta$ where $z \in F$ then $\exists\ x \in E$ with $\|x\| \le 1$ so that $Ax = z$. By homogeneity,

i.e. $z = Ax$ for some $x \in E$ with

$$\|x\| \le \frac{\|z\|}{\delta}$$

Fix $z \in F$. I want to find $x \in E$
so that $(A+B)x = z$.

Since A is onto, $\exists\, x_0 \in E$ so that
$Ax = z$ and $\|x\| \le \frac{\|z\|}{\delta}$.

i.e. $\delta \|x_0\| \le \|z\| = \|Ax_0\| \le \|A\|\,\|x_0\|$

I use x_0 to define $Bx_0 \in F$.

Since A is onto, $\exists\, x_1$ with $Ax_1 = -Bx_0$

and $\delta \|x_1\| \le \|-Bx_0\| \le \|B\|\,\|x_0\|$

i.e. $\|x_1\| \le \frac{\|B\|}{\delta} \|x_0\|$

I use x_1 to define $Bx_1 \in F$ and $\exists\, x_2$
so that $Ax_2 = -Bx_1$ and

$$\delta \|x_2\| \le \|-Bx_1\| \le \|B\|\,\|x_1\|$$

$$\Rightarrow \|x_2\| \le \left(\frac{\|B\|}{\delta}\right)^2 \|x_0\|$$

I proceed inductively. $Ax_n = -Bx_{n-1}$
and $\|x_n\| \le \left(\frac{\|B\|}{\delta}\right)^n \|x_0\|$.

Now

$$(A+B)\sum_0^n x_n = Ax_0 + \sum_1^n Ax_n + \sum_0^{n-1} Bx_n + Bx_n$$

$$= Ax_0 + Bx_n$$

Also, if I choose $\|B\| < \delta$ (this is my "B is small" requirement)

then $\|x_n\| \to 0$ as $n \to \infty$.

Therefore if $\sum_0^n x_n$ converges to x_∞ then

$$(A+B)[x_\infty] = \lim_{n\to\infty} (A+B)\sum_0^n x_n = \lim_{n\to\infty} Ax_0 + Bx_n$$

$$= Ax_0 = z$$ and we've shown $A+B$ is onto!

So it suffices to show $\sum_0^n x_n$ converges. Since E is a Banach space, it suffices to show that $\sum_0^\infty \|x_n\|$ converges. This follows immediately since

$$\sum_0^n \|x_n\| \le \sum_0^n \left(\frac{\|B\|}{\delta}\right)^n \|x_0\|$$

$$\le \|x_0\| \frac{1}{1-\|B\|/\delta}$$

$\Rightarrow \sum_0^n \|x_n\|$ converges as $n \to \infty$. ✓ //

Problem 4: Let $A \in \mathcal{L}(E, E)$.

Prove that $R = \lim_{n \to \infty} \sqrt[n]{\|A^n\|}$ exists and

show that $\lambda \in \text{spectrum}(A) \Rightarrow |\lambda| \le R$.

proof: See the lecture notes of Feb 12 for the spectral radius theorem.

Problem 5: Let $E = C([0,1])$ with L^∞ metric.

define $V : E \to E$ by $Vx(t) := \int_0^t x(s)\, ds$.

prove $V \in \mathcal{L}(E,E)$ and $\text{Spectrum}\{V\} = \{0\}$

1) $V(x+y)(t) = \int_0^t (x+y)(s)\, ds = \int_0^t x(s) + y(s)\, ds$

$= \int_0^t x(s)\, ds + \int_0^t y(s)\, ds$

$= Vx(t) + Vy(t)$ so V is linear.

$$\|Vx\| = \sup_{t \in [0,1]} |Vx(t)| = \sup_{t \in [0,1]} \left| \int_0^t x(s)\, ds \right|$$

$$\le \sup \int_0^t |x(s)|\, ds \le \sup_{t \in [0,1]} \|x\| \int_0^t ds$$

$$= \|x\| \sup_{t \in [0,1]} t = \|x\|.$$ $\Rightarrow$ V is bounded linear operator.

2) I'll use the spectral radius theorem to control the spectrum of V.

$$V^2 x(t) = V(Vx)(t)$$

$$= \int_0^t Vx(s)\,ds = \int_0^t\int_0^s x(\tau)\,d\tau\,ds$$

$$= \int_0^t (t-s)x(s)\,ds \qquad \text{(integration by parts.)}$$

$$V^3 x(t) = \int_0^t 1\cdot\int_0^s (s-s')x(s')\,ds'\,ds$$

$$= \int_0^t (t-s)\int_0^s x(s')\,ds'\,ds$$

$$= \int_0^t \frac{(t-s)^2}{2}\,x(s)\,ds$$

In general, $V^n x[t] = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\,x(s)\,ds$

assume true for n, prove for $(n+1)$

$$V^{n+1}x[t] = \int_0^t V^n x[s]\,ds = \int_0^t 1\cdot\int_0^s \frac{(s-s')^{n-1}}{(n-1)!}\,x(s')\,ds'\,ds$$

$$= \int_0^t (t-s)\int_0^s \frac{(s-s')^{n-2}}{(n-2)!}\,x(s')\,ds'\,ds$$

$$= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\int_0^s x(s')\,ds'\,ds = \int_0^t \frac{(t-s)^n}{n!}\,x(s)\,ds$$

$$\|V^n x\| = \sup_{t\in[0,1]} \left| \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} x(s)\, ds \right|$$

$$\leq \frac{1}{(n-1)!} \|x\|$$

$$\Rightarrow \quad \|V^n\| \leq \frac{1}{(n-1)!}$$

$$\Rightarrow \quad \|V^n\|^{1/n} \leq \frac{1}{((n-1)!)^{1/n}}$$

Stirling's approximation: $n! \sim C \dfrac{n^n \sqrt{n}}{e^n}$

$$\left[(n-1)!\right]^{\frac{1}{n}} = \left[\frac{1}{n}\cdot n!\right]^{\frac{1}{n}} \sim \frac{n}{e}\left(\frac{1}{\sqrt{n}}\right)^{1/n}$$

$$\frac{n}{e} \to \infty \quad \text{as} \quad n \to \infty$$

$$\left(\frac{1}{\sqrt{n}}\right)^{1/n} \to 1 \quad \text{as} \quad n \to \infty$$

$$\therefore \lim_{n\to\infty} \|V^n\|^{1/n} \leq \lim_{n\to\infty} \frac{C}{\frac{n}{e}\left(\frac{1}{\sqrt{n}}\right)^{1/n}} = 0$$

$\Rightarrow$ spectrum$(V) = \{0\}$. //