Solutions to fifth HW assignment.

K+F p128 #6

assume $f, f_1 \ldots f_n$ are linear functionals on L so that

$$f_1(x) = f_2(x) = \cdots = f_n(x) \Rightarrow f(x) = 0.$$

Show that $\exists$ constants $a_1 \ldots a_n$ so that

$$f = \sum_1^n a_i f_i$$

proof: I'll do this by showing that I can actually write f as a linear combination of a subset of $\{f_i\}_1^n$

1) if $L_{f_{i_0}} = L$ then throw f_{i_0} out since $f_{i_0} \equiv 0$ and is useless.

2) if $L_{f_{i_0}} = L_{f_{j_0}}$ for some i_0, j_0 then we know that f_{i_0} is a multiple of f_{j_0}, $\Rightarrow$ we don't need both of them $\Rightarrow$ throw out f_{j_0} (if $i_0 < j_0$, say)

3) if $\bigcap_1^j L_{f_i} = L_{f_{j+1}}$ then throw out f_{j+1}.

Now that we've culled the set

$\{f_1, \dots f_n\}$,

we have a set $\{f_1 .. f_{\tilde{n}}\}$ where $\tilde{n} \leq n$ and I've renumbered the functions.

Note that $\displaystyle\bigcap_1^n L_{f_i} = \bigcap_1^{\tilde{n}} L_{f_i}$

intersection of null spaces of original set of functions

intersection of null spaces of culled set of functions.

Now I'll construct my linear combination.

Let $x \in L$. Then $\exists\, x_1 \in L$ s.t. $x_1 \notin L_{f_1}$ and

$$x = \alpha_1 x_1 + w \quad \text{where } w \in L_{f_1}.$$

Since $L_{f_2} \neq \bigcap_1^1 L_{f_i}$, $\exists\, x_2 \in L_{f_1}$ $x_2 \notin L_{f_2}$

and $w = \alpha_2 x_2 + z$ where $z \in L_{f_2}$. But since $w \in L_{f_1}$ and $x_2 \in L_{f_1}$, we see $z \in L_{f_1} \cap L_{f_2}$.

$$\Rightarrow \quad x_1 = \alpha_1 x_1 + \alpha_2 x_2 + z \quad \text{where } z \in L_{f_1} \cap L_{f_2}$$

Since $L_{f_3} \neq L_{f_1} \cap L_{f_2}$, $\exists\, x_3 \in L_{f_1} \cap L_{f_2}$, $x_3 \notin L_{f_3}$

$\Rightarrow \quad z = \alpha_3 x_3 + \tilde{w}$ where $\tilde{w} \in L_{f_3}$. But since $z \in L_{f_1} \cap L_{f_2}$ and $x_3 \in L_{f_1} \cap L_{f_2}$, we see $\tilde{w} \in L_{f_1} \cap L_{f_2} \cap L_{f_3}$

Proceeding in this way, we write

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_{\tilde n} x_{\tilde n} + k$$

where $\quad x_j \in \bigcap_1^{j-1} Lf_i \quad$ for $\quad 2 \dots \tilde n$

and $\quad f_j(x_j) \neq 0 \quad$ and $\quad k \in \bigcap_1^{\tilde n} Lf_i$

now
$$f_1(x) = \alpha_1 f_1(x_1)$$
$$f_2(x) = \alpha_1 f_2(x_1) + \alpha_2 f_2(x_2)$$
$$\vdots$$
$$f_{\tilde n}(x) = \alpha_1 f_n(x_1) + \alpha_2 f_n(x_2) + \dots + \alpha_{\tilde n} f_{\tilde n}(x_{\tilde n})$$

$$\Rightarrow \quad A \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{\tilde n} \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \\ f_{\tilde n}(x) \end{pmatrix}$$

where A is invertible since it's lower triangular and all the diagonal entries are nonzero.

finally, $\quad f(x) = \alpha_1 f(x_1) + \alpha_2 f(x_2) + \dots + \alpha_{\tilde n} f(x_{\tilde n}) + f(k)$

since $\quad f(k) = 0, \quad$ we have

$$f = \sum_1^{\tilde n} b_i f_i(x)$$

where the b_i depend on the $f(x_1), f(x_2) \dots f(x_{\tilde n})$ and $\quad f_i(x_j) \quad i \leq j.$

K+F p 137 #1.

$M \subset l^2(\mathbb{R}, \mathbb{N})$

$$M = \{x \mid \sum_1^\infty n^2 x_n^2 \le 1\}$$

claim: M is a convex set.

proof: Let $x, y \in M$ $t \in (0,1)$.

$z = tx + (1-t)y$

$$\sum_1^\infty n^2 [t x_n + (1-t) y_n]^2 = \sum_1^\infty n^2 t^2 x_n^2 + n^2\, 2t(1-t) x_n y_n + n^2 (1-t)^2 y_n^2$$

$$= t^2 \sum_1^\infty n^2 x_n^2 + 2t(1-t) \sum_1^\infty n^2 x_n y_n + (1-t)^2 \sum_1^\infty n^2 y_n^2$$

$$\le t^2 \cdot 1 + (1-t)^2 \cdot 1 + 2t(1-t) \sqrt{\sum_1^\infty n^2 x_n^2} \sqrt{\sum_1^\infty n^2 y_n^2}$$

by Cauchy-Schwartz

$$\le t^2 + (1-t)^2 + 2t(1-t)$$

$$= (t + (1-t))^2 = 1$$

$\Rightarrow \sum n^2 z_n^2 \le 1 \Rightarrow z \in M \Rightarrow M$ is convex.

!!Note!! I used $t(1-t) \ge 0$ in the Cauchy-Schwartz step!

<u>claim</u>: M is not a convex body.

<u>proof</u>: let $x \in M$ and $y \in l^2(\mathbb{R}, \mathbb{N})$. I want to find $\varepsilon_y > 0$ so that $|t| < \varepsilon_y$ $\Rightarrow$ $x + ty \in M$. Take $y_k = \frac{1}{k}$. then $y \in l^2(\mathbb{R}, \mathbb{N})$ but $\sum n^2 y_n^2 \not< \infty$. $\Rightarrow$ $x + ty \in M \Leftrightarrow t = 0$. $\Rightarrow$ M is not a convex body.

K+F, page 137, #2

Give an example of two convex bodies whose intersection is not a convex body

Look in $\mathbb{R}$.

$$M = [0, 1), \quad N = [1, 2]$$

both are convex bodies.

$M \cap N = \{1\}$ which is convex but isn't a convex body.

K+F, page 137, #3

$x_1 \dots x_{n+1}$ in L are in general position if $\nexists$ $(n-1)$-dimensional subspace that contains $x_1 \dots x_{n+1}$.

the convex hull of ~~n~~ n+1 points in general position is called an n-dimensional simplex and $x_1 \ldots x_{n+1}$ are the vertices of the simplex. $co(x_1, \ldots x_{n+1})$ = convex hull

1) describe the 0-d, 1-d, 2-d, and 3-d simplices in $\mathbb{R}^3$. Okay... this is a little vague/annoying.

0-d $\Rightarrow$ $\{x_1\}$ and $co(x_1) \not\subset$ (-1)-dimensional subspace of $\mathbb{R}^3$

This will be true for all $\{x\} \subset L$ (since I'll assume something sensible...).

$\boxed{\text{the 0-d simplices in } \mathbb{R}^3 \text{ are isolated points.}}$

1-d simplex $\Rightarrow$ $\{x_1, x_2\}$ and $co(x_1, x_2) \not\subset$ 0-dim subspace of $\mathbb{R}^3$

i.e. $co(x_1, x_2) \neq \vec{0}$

the 1-d simplices are 1) a single isolated ~~points~~ point that isn't ~~aren't~~ $\vec{0}$ and 2) any line segment.

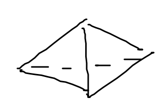
2-d simplex $\Rightarrow$ $\{x_1, x_2, x_3\}$ and $co(x_1, x_2, x_3) \not\subset$ 1-d subspace of $\mathbb{R}^3$

i.e. $co(x_1, x_2, x_3) \not\subset$ line through $\vec{0}$.

the 2-d simplices are planar 3-sided convex polyhedra and line segments that don't lie on a line through $\vec{o}$.

3-d simplex $\Rightarrow \{x_1, x_2, x_3, x_4\}$ and $co(x_1, .. x_4) \not\subset$ 2-d subspace of $\mathbb{R}^4$.

i.e planar 3 and 4 sided objects that lie in a plane that doesn't contain $\vec{o}$ and solid polyhedra that have nontrivial interiors and 2-d faces



(the K+F definition allows for a certain degree of degeneracy --- a simplex can have a vertex which is not an extreme point.)

2) Prove that the simplex w/ vertices $x_1 .. x_{n+1}$ is the set of points in L that can be written as

$$x = \sum_1^{n+1} \alpha_k x_k \quad \text{where } \alpha_k \geq 0, \ \sum_1^{n+1} \alpha_k = 1.$$

Let $M = \{ \sum_1^{n+1} \alpha_k x_k \mid \alpha_k \geq 0 \ \sum_1^{n+1} \alpha_k = 1 \}$

first, I claim that

$$co(x_1 .. x_{n+1}) \subseteq M.$$

to prove this, it suffices to show that

$x_i \in M$ for each i.

and M is convex. Since $co(x_1 .. x_{n+1})$ is the smallest convex set containing $x_1 .. x_{n+1} \Rightarrow co(x_1 .. x_{n+1}) \subseteq M$.

first, $x_i \in M$ since I can take $\alpha_i = 1$ $\alpha_j = 0$ $j \neq i$

then $x_i = \sum \alpha_k x_k \in M$.

Let $x, y \in M$, $t \in (0,1)$

then $tx + (1-t) y = t \sum \alpha_k x_k + (1-t) \sum \beta_k x_k$

$$= \sum (t\alpha_k + (1-t) \beta_k) x_k$$

$$= \sum \gamma_k x_k.$$

since $\alpha_k, \beta_k \geq 0$ and $t \in (0,1)$ we have $\gamma_k \geq 0$.

also $\sum \gamma_k = t \sum \alpha_k + (1-t) \sum \beta_k = t + (1-t) = 1$ ✓

$\Rightarrow M$ is convex and $co(x_1 .. x_{n+1}) \subseteq M$.

Now we want to prove $M \subseteq co(x_1 ... x_{n+1})$.

$S_1 = \{x_1\}$ $\quad S_2 = \alpha_2 x_2 + (1-\alpha_2) x_1 \quad 0 \leq \alpha_2 \leq 1$

by construction, $S_2 \subseteq co(x_1, x_2) \subseteq co(x_1 .. x_{n+1})$

$S_3 = \alpha_3 X_3 + (1-\alpha_3) y$ where $y \in S_2$ $\quad 0 \le \alpha_3 \le 1$

by construction, $S_3 \subseteq Co(X_1, X_2, X_3) \subseteq Co(X_1, \ldots X_{n+1})$

and $y = \alpha_1 X_1 + \alpha_2 X_2$ where $\alpha_1 + \alpha_2 = 1$ $\quad \alpha_1, \alpha_2 \geq 0$

$\Rightarrow$ $\alpha_3 X_3 + (1-\alpha_3)\alpha_1 X_1 + (1-\alpha_3)\alpha_2 X_2$

$= \sum_1^3 \beta_i X_i$ where $\quad \beta_i \geq 0$

$\sum_1^3 \beta_i = \alpha_3 + (1-\alpha_3)(\alpha_1 + \alpha_2) = \alpha_3 + (1-\alpha_3)\,1 = 1.$

continuing in this way,

$S_k = \{ \sum_1^k \alpha_k X_k \mid \alpha_k \geq 0, \sum_1^k \alpha_k = 1 \} \subset Co(X_1 \ldots X_{n+1})$

and

$S_{k+1} = \alpha_{k+1} X_k + (1-\alpha_{k+1}) y$ $\quad$ where $y \in S_k$

$= \sum_1^{k+1} \beta_i X_i$ $\quad \beta_i \geq 0$ and

$\sum_1^{k+1} \beta_i = \alpha_{k+1} + (1-\alpha_{k+1}) \sum_1^k \gamma_k = \alpha_{k+1} + (1-\alpha_{k+1}) = 1$

Hence

$S_1 \subset S_2 \subset \ldots \subset S_{n+1} \subset Co(X_1 \ldots X_{n+1})$

where $S_{n+1} = \sum_1^{n+1} \gamma_i X_i$ $\quad$ with $\quad \gamma_i \geq 0$ $\quad \sum_1^{n+1} \gamma_i = 1$ ✓

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K+F, page 137, #4.

Claim: if $x_1 \ldots x_{n+1}$ are in general position then so is any subset of $k+1$ of them $(k<n)$.

Proof: assume not. then $\exists\, k_0 < n$ and a subset of k_0+1 of the points so that

$$x_{i_1}, x_{i_2}, \ldots x_{i_{k_0+1}} \subset L_0$$

where L_0 is a k_0-1 dimensional subspace of L.

renumber the x_i's so $x_1 \ldots x_{k+1}$ are the vectors in question.

$\Rightarrow \exists\, k_0-1$ linearly independent elements of L, $y_1 \ldots y_{k_0-1}$ so that

$$x_1 = \sum_1^{k_0-1} \alpha_{1i} y_i \quad \ldots \quad x_{k_0+1} = \sum_1^{k_0-1} \alpha_{k_0+1, i} y_i$$

Let L_1 be the smallest subspace that contains $y_1 \ldots y_{k_0-1}$ and $\underbrace{x_{k_0+2} \ldots x_{n+1}}_{n-k_0 \text{ of these}}$

then $\dim(L_1) \le k_0 - 1 + n - k_0 = n-1$ and this shows that $x_1 \ldots x_{n+1}$ aren't in general position. //