Fourth HW assignment

1a) assume f is upper + lower semicontinuous. Prove f is continuous.

proof: let $U \subset \mathbb{R}$ be open.

If $f^{-1}(U) = \emptyset$ then we're done since $\emptyset$ is open.
Let $x \in f^{-1}(U)$, then $f(x) \in U$ and $\exists\, \varepsilon > 0$ so that $(x-\varepsilon, x+\varepsilon) \subset U$. $\Rightarrow$ $f^{-1}((x-\varepsilon, x+\varepsilon)) \subset f^{-1}(U)$.
Since $f^{-1}((x-\varepsilon, x+\varepsilon)) = f^{-1}((-\infty, x+\varepsilon)) \cap f^{-1}((x-\varepsilon, \infty))$ and we know those sets are open (using the upper + lower semicont.) we know $\exists$ open set containing x that is contained in $f^{-1}(U)$. Call $V_x = f^{-1}((x-\varepsilon, x+\varepsilon))$. In this way, we show $f^{-1}(U) = \bigcup_x V_x$ a union of open sets and so $f^{-1}(U)$ is open. ✓

1b) Assume f & g are lower semicontinuous. Prove that $f+g$ and $f \vee g$ is lower semicont.
Fix $\beta \in \mathbb{R}$. then

$$(f+g)^{-1}((\beta,\infty)) = \bigcup_{\alpha \in \mathbb{R}} \overbrace{f^{-1}((\alpha,\infty))}^{\swarrow \text{open}} \cap \overbrace{g^{-1}((\beta-\alpha,\infty))}^{\swarrow \text{open}}$$

$\Rightarrow (f+g)^{-1}((\beta,\infty))$ is the union of open sets and is open.

$(f \vee g)^{-1}((\beta, \infty)) = f^{-1}((\beta,\infty)) \cup g^{-1}((\beta,\infty))$
is the union of open sets $\Rightarrow$ is open.
thus $f \vee g$ is lower semi cont. //

1c) Let $\{f_n\}$ be lower semi, Define
$$f(x) = \sup_n f_n(x).$$
Show f is lower semicontinuous.

proof: Want $f^{-1}((\beta,\infty))$ open.
Assume $f(x) \in (\beta,\infty)$. Then given $\varepsilon > 0$ $\exists\, n_\varepsilon \ni$
$$\beta < f(x) - \varepsilon < f_{n_\varepsilon}(x) \leq f(x).$$
$\Rightarrow x \in f_{n_\varepsilon}^{-1}((\beta,\infty)) \leftarrow$ open set.

$$\Rightarrow f^{-1}((\beta,\infty)) \subseteq \bigcup_n f_n^{-1}((\beta,\infty)).$$

Now claim $\bigcup_n f_n^{-1}((\beta,\infty)) \subseteq f^{-1}((\beta,\infty))$.
Assume not. Let $x \in \bigcup_n f_n^{-1}((\beta,\infty))$ and $x \notin f^{-1}((\beta,\infty))$
$\Rightarrow f_{n_0}(x) > \beta$ for some n_0. but $f_n(x) \leq \beta$
This is impossible because $f(x) \geq f_n(x)$ $\forall n$.
thus $f^{-1}((\beta,\infty)) = \bigcup_n f_n^{-1}((\beta,\infty))$
= union of opensets $\Rightarrow$ is open
$\Rightarrow f$ is lower semi cont. //

1d) let ϕ be a step function defined on the partition

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

<u>claim</u>: ϕ is lower semicontinuous

$\iff \phi(x_i) \leq \min\{\phi \text{ on } (x_{i-1}, x_i) \text{ and } \phi \text{ on } (x_i, x_{i+1})\}$

<u>proof</u>: Denote the values of ϕ in (x_{i-1}, x_i) by y_i.

$\Rightarrow y_1, y_2, \ldots y_n \in \text{range } \phi$.

($\Rightarrow$) assume $\phi(x_{i_0}) > \min\{y_{i_0}, y_{i_0+1}\}$ some i_0.

I'll show ϕ is not lower semicontinuous.

let $\beta = \dfrac{\phi(x_{i_0}) + \min\{y_{i_0}, y_{i_0+1}\}}{2}$

then $\phi^{-1}((\beta, \infty))$ contains x_{i_0} and excludes (x_{i_0-1}, x_i) or (x_{i_0}, x_{i_0+1}) $\Rightarrow \phi^{-1}((\beta, \infty))$ is not open since $\nexists$ neighborhood of x_{i_0} that's contained in $\phi^{-1}((\beta, \infty))$.

($\Leftarrow$) Assume $\phi(x_i) \leq \min\{y_i, y_{i+1}\}$. Since ϕ is a step function, its range is discrete:

$$\{\phi(x_i) \mid 0 \leq i \leq n\} \cup \{y_i \mid 1 \leq i \leq n\}$$

fix β. if $x_{i_0} \in \phi^{-1}((\beta, \infty))$ then

(x_{i_0-1}, x_{i_0}) and $(x_{i_0}, x_{i_0+1}) \in \phi^{-1}((\beta, \infty))$ since $\phi(x_{i_0}) \leq y_i, y_{i+1}$

$\Rightarrow x_{i_0}$ is not isolated in $\phi^{-1}((\beta, \infty))$.

$\Rightarrow \phi^{-1}((\beta, \infty)) =$ union of open intervals $\Rightarrow \phi$ is lower semi. //

1e) <u>claim</u>: $f : [a,b] \to \mathbb{R}$ is lower semicont if and only if $\exists$ monotone increasing sequence of lower semicontinuous step functions $\{\phi_n\}$ such that for each $x \in [a,b]$, $f(x) = \lim_{n \to \infty} \phi_n(x)$.

<u>proof</u>:

($\Leftarrow$) since $\phi_n(x) \uparrow f(x)$, we have $f(x) = \sup_n \phi_n(x)$. $\Rightarrow f$ is lower semicont by part c.

($\Rightarrow$) Assume f is lower semicontinuous. First I'll construct a sequence of partitions

$$a = x_0 < x_1 = b \qquad \underline{0th}$$

$$a = x_0 < x_1 < x_2 = b \qquad \underline{1st}$$

$$a = x_0 < x_1 < x_2 < x_3 < x_4 = b \qquad \underline{2nd}$$

etc where the subintervals are of equal length: $\frac{b-a}{2^n}$ is the length of the subintervals in the $n\underline{th}$ partition. If x_0 is a meshpoint for the $n\underline{th}$ partition ($x_0 = a + j\frac{b-a}{2^n}$) then x_0 is a meshpoint for the $m\underline{th}$ partition if $m > n$. Since

$$x_0 = a + J\frac{b-a}{2^{m+n}} \qquad \text{where } J = j2^{m-n}$$

Now, I define my step functions.

$$\phi_n(x) = \min_{x \in [x_i, x_{i+1}]} f(x) \quad \text{if } x \in (x_i, x_{i+1}).$$

$\Rightarrow$ $\phi_n \leq f$ for all $x \in \bigcup_i^n (x_{i-1}, x_i)$

Now, we define ϕ_n at the meshpoints by

$$\phi_n(x_i) = \min\{y_i, y_{i+1}\}$$

where $y_i = \phi_n$ on (x_{i-1}, x_i), $y_{i+1} = \phi_n$ on (x_i, x_{i+1}).

ϕ_n is lowersemicontinuous by part (1d).

Also, since I defined y_i by taking the min over $[x_{i-1}, x_i]$ rather than the inf over (x_{i-1}, x_i), I know that $\phi_n \leq f$ on $[a,b]$.

(Since $\phi_n(x_i) \leq f(x_i)$ by constr.)

Also, if $m > n$ then $\phi_m \geq \phi_n$ on $[a,b]$.

$\Rightarrow$ $\{\phi_n\}$ is a monotone increasing family of step functions such that $\phi_n \leq f$ for each n.

<u>Claim</u>: $f(x) = \lim_{n\to\infty} \phi_n(x) \quad \forall x \in [a,b]$.

assume not. Let $x \in [a,b]$ be a point such that $\lim_{n\to\infty} \phi_n(x) < f(x)$.

fix β so that $\lim_{n\to\infty} \phi_n(x) < \beta < f(x)$.

if x is never a partition point then do the following. Fix n. then $x \in (x_{i-1}, x_i)$ some i. and $\tilde{x}_n \in [x_{i-1}, x_i]$ so that $\phi_n(x) = f(\tilde{x}_n)$

$\Rightarrow \lim_{n\to\infty} \phi_n(x) = \lim_{n\to\infty} f(\tilde{x}_n) < \beta$

where $\tilde{x}_n \to x$ by construction. $\Rightarrow f^{-1}((\beta,\infty))$ is not open because $x \in f^{-1}((\beta,\infty))$ but we cannot find a neighborhood containing x that is contained in $f^{-1}((\beta,\infty))$.

if x is a partition point for some N, then x is a partition point $\forall m \geq N$ Fix $m \geq N$. then $\exists \tilde{x}_m \in [x_{i-1}, x_{i+1}]$ (where $x = x_i$) so that $\phi_m(x) = f(\tilde{x}_m)$. Now repeat argument as above.

this shows that if f is lower semicont then $f = \lim_{n\to\infty} \phi_n$ and we're done //

Problem 2

part a) prove the proposition.

Proposition part 1

define τ' on Y by $U \in \tau' \Leftrightarrow p^{-1}(U) \in \tau$ where τ is the topology on X. We want to show τ' is a topology.

$Y \in \tau'$ since $p^{-1}(Y) = X \in \tau$

$\phi \in \tau'$ since $p^{-1}(\phi) = \phi \in \tau$

assume $U_\alpha \in \tau'$. Then

$$p^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha p^{-1}(U_\alpha) \in \tau \text{ since}$$

each $p^{-1}(U_\alpha) \in \tau$ and τ is a top.

$$\Rightarrow \bigcup_\alpha U_\alpha \in \tau'.$$

assume $U_1, U_2 \in \tau'$,

then $p^{-1}(U_1), p^{-1}(U_2) \in \tau$ and

$$p^{-1}(U_1 \cap U_2) = p^{-1}(U_1) \cap p^{-1}(U_2) \in \tau$$

since τ is a topology

$$\Rightarrow U_1 \cap U_2 \in \tau'.$$

this proves τ' is a topology on Y

also p is continuous from (X, τ) to (Y, τ') by the construction of τ'.

//

proof of proposition part 2

$U \in \mathcal{G}^0 \Leftrightarrow U = f^{-1}(V)$ for some $V \in \tau'$
some $f \in \mathcal{F}$.

$\mathcal{G} :=$ all finite intersections of members of $\mathcal{G}^0$.

we want to prove $\mathcal{G}^0$ is a subbase for τ.
So we need to prove

1) given $x \in X$ $\exists$ $G \in \mathcal{G}$ with $x \in G$
2) $G_\alpha, G_\beta \in \mathcal{G}$ and $x \in G_\alpha \cap G_\beta$ then $\exists$ $G_\gamma \in \mathcal{G}$ with $x \in G_\gamma \subset (G_\alpha \cap G_\beta)$

first, if $x \in X$ then $x \in f^{-1}(Y) = X$ for any $f \in \mathcal{F}$.
$\Rightarrow$ since $X \in \mathcal{G}^0$, we've found a member of $\mathcal{G}$ that contains $\mathcal{G}^0$. (since $\mathcal{G}^0 \subset \mathcal{G}$.)

second, let $G_\alpha, G_\beta \in \mathcal{G}$ then $G_\alpha = \bigcap_1^n U_i$ some $U_i \in \mathcal{G}^0$ and $G_\beta = \bigcap_1^m U_j$ some $U_j \in \mathcal{G}^0$. $\Rightarrow$ $G_\alpha \cap G_\beta = \bigcap_1^n \bigcap_1^m U_i$
(sorry! hideous abuse of notation!) and $G_\alpha \cap G_\beta \in \mathcal{G}$.
this proves $\mathcal{G}^0$ is a subbase

Finally, we have to prove that each $f \in \mathcal{F}$ is continuous with respect to this topology. let $V \in \tau'$.
then $f^{-1}(V) \in \mathcal{G}^0 \subset \mathcal{G} \subset \tau$. $\Rightarrow$ $f^{-1}(V)$ is open
$\Rightarrow$ f is cts. //

2b) let $X = \mathbb{R}^2$ and let $(Y, \mathcal{T}')$ be $\mathbb{R}$ with the usual metric topology.
for each $\vec{a} \in X$ define

$$f_{\vec{a}} : X \to Y \quad \text{by} \quad f_{\vec{a}}(\vec{x}) = \langle \vec{a}, \vec{x} \rangle$$

let $\mathcal{F} = \{ f_{\vec{a}} \mid \vec{a} \in \mathbb{R}^2 \}$ and define $\mathcal{T}_W$ a topology on $\mathbb{R}^2$ as in part 2 of proposition.

<u>claim</u>: $\mathcal{T}_W$ = usual metric topology on $\mathbb{R}^2$

<u>proof</u>: I'll do this by using the theorem that says:

$\mathcal{G}$ is a base for $\mathcal{T} \Leftrightarrow$ given $U \in \mathcal{T}$ and $x \in U$, $\exists\, G \in \mathcal{G}$ with $x \in G \subset U$.

Let $\mathcal{T}_\infty$ = metric topology on $\mathbb{R}^2$ based on l^∞ metric.

$\mathcal{G}_\infty$ is the base of l^∞ open balls
$\mathcal{G}_W$ is the base described in prop. (called $\mathcal{G}$ there)

(*) If $U \in \mathcal{T}_\infty$, $x \in U$ and $\exists\, G_W \in \mathcal{G}_W$ with $x \in G_W \subset U$ then $\mathcal{G}_W$ is a base for $\mathcal{T}_\infty$

(**) If $U \in \mathcal{T}_W$, $x \in U$ and $\exists\, G_\infty \in \mathcal{G}_\infty$ with $x \in G_\infty \subset U$ then $\mathcal{G}_\infty$ is a base for $\mathcal{T}_W$

If both of these are true then

$$\mathcal{T}_\infty \subset \mathcal{T}_W \subset \mathcal{T}_\infty \Rightarrow \mathcal{T}_\infty = \mathcal{T}_W \quad \text{as desired}$$

And so it remains to prove (*) and (**)

(*) Let $U \in \mathcal{T}_\infty$, $\vec{x} \in U$. Then $\exists\ \varepsilon > 0$ so that $S(x, \varepsilon) \subset U$. $\vec{x} = (x_1, x_2)^T$

Take $\vec{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$U_1 = f_{\vec{a}_1}^{-1}((-\infty, x_1 + \varepsilon))$$

$$U_2 = f_{\vec{a}_1}^{-1}((x_1 - \varepsilon, \infty))$$

Take $\vec{a}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$U_3 = f_{\vec{a}_2}^{-1}((-\infty, x_2 + \varepsilon))$$

$$U_4 = f_{\vec{a}_2}^{-1}((x_2 - \varepsilon, \infty))$$

Then $U_1 \cap U_2 \cap U_3 \cup U_4 \in \mathcal{G}_w$ and

$$U_1 \cap U_2 \cap U_3 \cap U_4 = S(x, \varepsilon) \subset U.$$

$\Rightarrow \mathcal{G}_w$ is a base for $\mathcal{T}_\infty$.

(**) Let $U \in \mathcal{T}_w$, $x \in U$. Then $\exists\ G_w \in \mathcal{G}_w$ with $x \in G_w \subset U$.

$$G_w = \bigcap_1^n f_{\vec{a}_j}^{-1}(V_j)$$

for some $\vec{a}_1, \ldots \vec{a}_n \subset \mathbb{R}^2$, $V_1, \ldots V_n \subset \mathcal{T}'$.

for each V_j, $\exists\ \varepsilon_j$ so that

$$\tilde{V}_j := S(f_{\vec{a}_j}(x), \varepsilon_j) \subset V_j.$$

$$\Rightarrow \quad \vec{x} \in \bigcap_{1}^{n} f_{\vec{a}_j}^{-1}(\tilde{V}_j) \subset G_w \in \mathcal{G}_w$$

Consider

$$f_{\vec{a}_j}^{-1}(\tilde{V}_j)$$

$\vec{a}_j = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ some a_1, a_2 and by defn,

$\tilde{x} \in f^{-1}_{\vec{a}_j}(\tilde{V}_j)$ iff

$$f_{\vec{a}_j}(\vec{x}) - \varepsilon_j < a_1(\tilde{x})_1 + a_2(\tilde{x})_2 < f_{\vec{a}_j}(\vec{x}) + \varepsilon_j$$

$$\Rightarrow \quad -\varepsilon_j < f_{\vec{a}_j}(\tilde{x} - \vec{x}) < \varepsilon_j$$

let $\delta_j < \dfrac{\varepsilon_j}{\sqrt{2}\sqrt{a_1^2 + a_2^2}}$

then you can check that

$$S(\vec{x}, \delta_j) \subset f_{\vec{a}_j}^{-1}(\tilde{V}_j)$$

Take $\delta = \min\{\delta_1, \dots \delta_n\}$ we have

$$S(\vec{x}, \delta) \subset f_{\vec{a}_j}^{-1}(\tilde{V}_j) \quad \forall j$$

$\Rightarrow \quad S(\vec{x}, \delta) \subset \bigcap_{1}^{n} f_{\vec{a}_j}^{-1}(\tilde{V}_j) \subset G_w$ as desired.

we've found a member of G_∞ that contains x and is contained in U. //

1c) from my proof of 1b) you see that I didn't need all of $\mathcal{F}$ to generate $\mathcal{T}_W$.
If I had the two functions

$$f_{\vec{a}_1} \text{ and } f_{\vec{a}_2} \qquad \vec{a}_1 = \begin{pmatrix}1\\0\end{pmatrix} \quad a_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

then they'd generate the entire topology.

Why? let $\mathcal{G}_{W_{12}}$ = the base generated using only $f_{\vec{a}_1}$ and $f_{\vec{a}_2}$.

by definition $\mathcal{G}_{W_{12}} \subset \mathcal{G}_W$

$\Rightarrow$ if $U \in \mathcal{T}_{W_{12}}$ $x \in U$ then $\exists\, G_{12} \in \mathcal{G}_{W_{12}}$ so that $x \in G_{12} \subset U$. But $G_{12} \in \mathcal{G}_W$ automatically so we've got one step.

if $U \in \mathcal{T}_W$ $x \in U$ then $\exists\, G \in \mathcal{G}_W$ so that $x \in G \subset U$. $\quad G = \bigcap_1^n f_{\vec{a}_j}^{-1}(V_j) \supset \bigcap_1^n f_{\vec{a}_j}^{-1}(\tilde{V}_j)$

as before and what I did before I found $S(x,\delta) \subset G$. But since $S(x,\delta) = U_1 \cap U_2 \cap U_3 \cap U_4$

(Where $U_1, .. U_4$ defined using $\vec{a}_1$ and $\vec{a}_2$ as in the proof of ⊛) I've found $x \in G_{W_{12}} \subset U$ as desired.

since I can take $G_{W_{12}} = U_1 \cap U_2 \cap U_3 \cap U_4$. //

1d) to generalize to $\mathbb{R}^n$, I'd just use the weak topology based on the unit vectors $\{\vec{e}_1 .. \vec{e}_n\}$

where $(\vec{e}_i)_j = \begin{matrix} 0 & i \neq j \\ 1 & i = j \end{matrix}$

In general, any set of n linearly independent vectors $\{\vec{a}_1, .. \vec{a}_n\}$ will generate τ_w.

<u>Problem 3</u>

a) prove $\{f_n\} \subset X$ converges weakly to f if and only if

$$\lim_{n \to \infty} \int_0^1 (f_n(x) - f(x))\, a(x)\, dx = 0$$

for all $a \in X$.

<u>proof</u>:

($\Longrightarrow$) assume f_n converges weakly to f.

fix $a \in X$ and $\varepsilon > 0$. We want to show $\exists N$ so that $n \geq N$

$$\Rightarrow \left| \int_0^1 (f_n(x) - f(x))\, a(x)\, dx \right| < \varepsilon$$

i.e. $\int f_n(x) a(x) dx \in \left(\int f(x) a(x) dx - \varepsilon, \int f(x) a(x) dx + \varepsilon \right)$

$=: V$

the set V is open in $\mathbb{R}$ and $f \in F_a^{-1}(V)$.

$F_a^{-1}(V)$ is open (is in τ_w) by definition of τ_w.

Since f_n converges weakly to f, $\exists$ N so that

$n \geq N \Rightarrow f_n \in F_a^{-1}(V) \Rightarrow n \geq N \Rightarrow |F_a(f_n) - F_a(f)| < \varepsilon$

as desired.

($\Leftarrow$) Assume $\lim_{n\to\infty} \int_0^1 (f_n(x) - f(x))\, a(x)\, dx = 0$ for all $a \in X$. Let $U \in \tau_w$ be a set that contains f. I want to show $\exists$ N so that $n \geq N \Rightarrow f_n \in U$.

$\exists$ $G \in \mathcal{G}_w$ with $f \in G \subset U$ since $\mathcal{G}_w$ is a base. by defn,

$$G = \bigcap_1^n F_{a_i}^{-1}(V_i) \quad \text{for some}$$

V_i open in $\mathbb{R}$, some $a_i \in X$.

Fix i. choose ε_i so that $S(F_{a_i}(f), \varepsilon_i) \subset V_i$

now let $\varepsilon = \min\{\varepsilon_i\}$. We know $\exists$ N_i so that $n \geq N_i \Rightarrow |F_{a_i}(f_n) - F_{a_i}(f)| < \varepsilon$

Take $N = \max\{N_i\}$. then $n \geq N \Rightarrow$

$$|F_{a_i}(f_n) - F_{a_i}(f)| < \varepsilon \quad \text{for each } i$$

$\Rightarrow n \geq N \Rightarrow f_n \in \bigcap_1^n F_{a_i}^{-1}(V_i) \subset U$ as desired. //

3b claim: if $f_n \to f$ in the metric topology then $f_n \to f$ in τ_w.

proof: by 3a) it suffices to show that for each $a \in X$ $\int_0^1 (f_n(x) - f(x)) a(x) dx \to 0$.

Let $a \in X$. then

$$\left| \int_0^1 (f_n - f)(a) dx \right| \leq \sqrt{\int (f_n - f)^2 dx} \sqrt{\int a^2 dx}$$

$\sqrt{\int a^2 dx}$ is some number, independent of n. Given $\varepsilon > 0$ $\exists N_\varepsilon$ so that

$$\sqrt{\int (f_n - f)^2 dx} < \frac{\varepsilon}{\sqrt{\int a^2 dx}}$$

if $n \geq N_\varepsilon$. (since $f_n \to f$ in metric topology)

$\Rightarrow \left| \int_0^1 (f_n - f) a \, dx \right| < \varepsilon$ as desired //

(note: I can argue similarly for L^p metric $1 \leq p \leq \infty$.)

3c) Let $f_n(x) = \cos(n \cdot 2\pi x)$.
I claim $f_n \to 0$ in τ_w but $f_n \not\to 0$ in the metric topology. First of all,

$$\rho_2(f_n, 0) = \sqrt{\int_0^1 \cos(n \cdot 2\pi x)^2 dx} = \frac{1}{\sqrt{2}}$$

so $f_n \not\to 0$ in metric topology.

Now I'll show $f_n \to 0$ in τ_w. By the results in 3c) it suffices to show

$$\int f_n(x)\, a(x) \to 0 \quad \forall a \in X.$$

Let $a \in X$.

$$\int_0^1 \cos(n\pi 2x)\, a(x)\,dx = -\int_0^1 \frac{1}{n \cdot 2\pi} \sin(n\pi 2x) \frac{da}{dx}(x)\,dx$$

$$\Rightarrow \left|\int_0^1 f_n(x)\, a(x)\,dx\right| = \frac{1}{n \cdot 2\pi} \left|\int_0^1 \sin(n\pi 2x) \frac{da}{dx}(x)\,dx\right|$$

$$\leq \frac{1}{n \cdot 2\pi} \underbrace{\int_0^1 \left|\frac{da}{dx}(x)\right| dx}_{\text{some number indep of } n}$$

$\Rightarrow$ for n big enough, $\left|\int_0^1 f_n(x)\, a(x)\right| < \varepsilon$.

But I have to be careful! I used that $\frac{da}{dx}$ exists! What if a is continuous and has no derivative? By Stone-Weierstrass, $\exists$ polynomial p so that $\max_{x \in [0,1]} |p(x) - a(x)| < \varepsilon/2$

and by the above, $\exists N$ so that $n \geq N$

$$\Rightarrow \left|\int f_n p\, dx\right| < \varepsilon/2$$

Now this gives

$$\left| \int_0^1 f_n(x)\, a(x)\, dx \right| \leq \left| \int_0^1 f_n(x)\, p(x)\, dx \right| + \left| \int_0^1 f_n(x) [a(x) - p(x)]\, dx \right|$$

$$< \varepsilon/2 + \int_0^1 |f_n(x)|\, |a(x) - p(x)|\, dx$$

$$< \varepsilon/2 + \varepsilon/2 \int_0^1 |f_n(x)|\, dx < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

so for $n \geq N$ we have $\left| \int f_n(x) a(x)\, dx \right| < \varepsilon$ as desired. //

This shows that its easier for a sequence to converge in τ_w than for it to converge in τ_{L^2}. So τ_w has "fewer" sets than τ_{L^2}.

Separately, one wants to ask whether

$$\tau_w \subset \tau_{L^2}.$$

<u>Problem 4</u> Ω = continuous real-valued functions on $[0, \infty)$ satisfying $\omega(0) = 0$

subbase $\mathcal{G}^0$ is given by

$$U(t_0, a, b) = \{ \omega \mid \omega(t_0) \in (a, b) \}$$

where $t_0 > 0$ and $-\infty \leq a < b \leq \infty$

4b)

$$A_{t_0} = \{ w \mid w(t_0) \neq 0 \}$$

is A_{t_0} open? closed?

$$U(t_0, 0, \infty) = \{ w \mid w(t_0) \in (0, \infty) \}$$

$$U(t_0, -\infty, 0) = \{ w \mid w(t_0) \in (-\infty, 0) \}$$

$$\Rightarrow A_{t_0} = \left(U(t_0, 0, \infty) \right) \cup \left(U(t_0, -\infty, 0) \right)$$

is open. ✓

For A_{t_0} to be closed we need

$$\{ w \mid w(t_0) = 0 \}$$

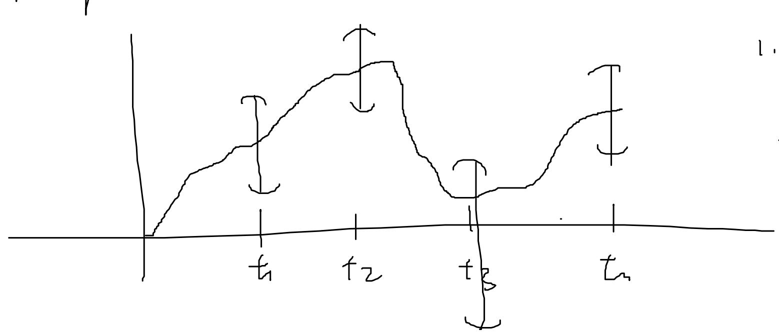
to be open. i.e. given $w_0 \in \{ w \mid w(t_0) = 0 \}$ we need to find $G \in \mathcal{G}$ so that

$$w_0 \in G \subset \{ w \mid w(t_0) = 0 \}.$$

if $G \in \mathcal{G}$ then

$$G = \bigcap_{i=1}^{n} U(t_i, a_i, b_i)$$

i.e. the paths that have $w(t_i) \in (a_i, b_i)$



i.e. for $w \in G$, it must pass through these windows at times t

whatever we choose as G, it cannot be contained in $\{ w \mid w(t_0) = 0 \}$

Why?

case 1: the window times include t_0.
then for $w_0 \in G$ we need $w(t_0) \in (a,b)$
where $0 \in (a,b)$. But just because
a path w has $w(t_0) \in (a,b)$ doesn't
mean that $w(t_0) = 0$! $\Rightarrow \exists\, w$ that
have $w \in G$ but $w(t_0) \neq 0$.

case 2: the window times don't include t_0
Then clearly $\exists\, w \in \Omega$ that are also
in G so that $w(t_0) \neq 0$ since G doesn't
even care about the time t_0.

$\Rightarrow$ It's impossible to find $G \in \mathcal{G}$
so that $w_0 \in G \subseteq \{w \mid w(t_0) = 0\}$
$\Rightarrow \{w \mid w(t_0) = 0\}$ is not open
$\Rightarrow \{w \mid w(t_0) \neq 0\}$ is not closed //

4a) verify that $\mathcal{G}^0 = \{ U(t,a,b) \mid t > 0,\ a, b \in [-\infty, \infty],\ a < b \}$
is a sub-base. To do this, I just need to show
that $\mathcal{G}$ = set of finite intersections satisfies
1) for each $w \in \Omega$ $\exists\, G \in \mathcal{G}$ with $w \in G$
2) if $G_\alpha, G_\beta \in \mathcal{G}$ and $w \in G_\alpha \cap G_\beta$ then $\exists\, G_\gamma \ni w \in G_\gamma \subset G_\alpha \cap G_\beta$

First, if $w \in \Omega$ then

$$w \in U(1, -\infty, \infty) \in \mathcal{G}^\circ \subset \mathcal{G}.$$

So the first requirement is done

Let $w \in G_\alpha \cap G_\beta$.

$$G_\alpha = \bigcap_1^n U(t_i, a_i, b_i) \qquad t_1 < t_2 < \cdots t_n$$

$$G_\beta = \bigcap_1^m U(s_j, a_j, b_j) \qquad s_1 < s_2 < \cdots s_m$$

take the union of the times. for each t in the union, if it occurs only in $\{t_1 .. t_n\}$ or only in $\{s_1 .. s_m\}$ then use (a_i, b_j) from G_α or G_β. If it occurs in both G_α or G_β's times then use

$$(a, b) = \underset{\text{from } G_\alpha}{(a_i, b_i)} \cap \underset{\text{from } G_\beta}{(a_j, b_j)}$$

we know the intersection is nonempty since $G_\alpha \cap G_\beta \neq \phi$.

In this way, we construct G_γ so that $G_\gamma = G_\alpha \cap G_\beta$

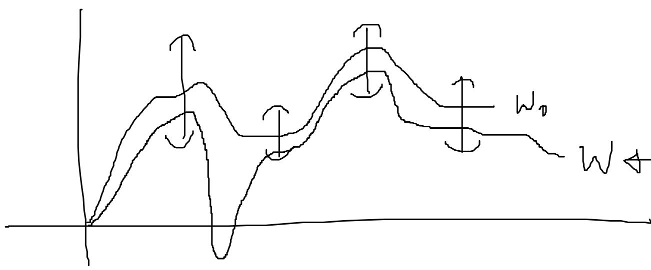
Which is more than what we needed since we just needed $w \in G_\gamma \subseteq G_\alpha \cap G_\beta$

thus $\mathcal{G}^\circ$ is a sub-base.

4c) Let $B_T = \{W \mid W(t) \neq 0 \quad \forall t \in (0,T)\}$

Is B_T open? closed?

Let $W_0 \in B_T$. If B_T were open, then $\exists\, G \in \mathcal{G}$ with $W_0 \in G \subset B_T$. But G has only finitely many windows and you have $w \in \Omega$ that satisfy $W \in G$ but $w(t_w) = 0$ some $t_w \in (0,T)$. $\Rightarrow$ $G \not\subset B_T \Rightarrow B_T$ is not open



Is B_T closed? If it were then its complement will be open.

$$\Omega - B_T = \{w \mid w(t_0) = 0 \text{ for some } t \in (0,T)\}.$$

If $W_0 \in \Omega - B_T$ then we need to find $G \in \mathcal{G}$ so that $W_0 \in G \subset (\Omega - B_T)$.

But from (4b) we see this will be impossible because however we choose G, we will be able to find $\tilde{w} \in G$ so that $\tilde{w} \neq 0$ for $t \in (0,T) \Rightarrow \tilde{w} \notin (\Omega - B_T)$.

$\Rightarrow B_T$ is not closed.

4d) is $C_T = \{\omega \mid \omega(t) = 0 \text{ for some } t \in (0,T]\}$ open? closed?

answer: $C_T = \Omega - B_T$. and B_T is neither open nor closed $\Rightarrow$ C_T is neither open nor closed //

Problem 5:

let (X, τ) be a topological space

U = a G_δ set if $U = \bigcap_1^\infty U_i$ where $U_i \in \tau$

U = a F_σ set if $U = \bigcup_1^\infty F_i$ where $X - F_i \in \tau$

claim: if U is G_δ then $X - U$ is F_σ

proof:
$$U = \bigcap_1^\infty U_i \Rightarrow X - U = X - \bigcap_1^\infty U_i$$
$$= \bigcup_1^\infty (X - U_i) = \bigcup_1^\infty F_i \text{ where } X - F_i \in \tau.$$

$\Rightarrow U$ is F_σ //

claim: if U is F_σ then $X - U$ is G_δ

proof:
$$U = \bigcup_1^\infty F_i \Rightarrow X - U = X - \bigcup_1^\infty F_i = \bigcap_1^\infty (X - F_i)$$
$$= \bigcap_1^\infty U_i \text{ where } U_i \in \tau.$$

$\Rightarrow X - U$ is G_δ. //

5b recall $B_T = \{ w \mid w(t) \neq 0 \text{ for } 0 < t < T \}$

prove that

B_T is a $G_{\delta\sigma\delta}$ set.

the first thing to realise is that since $(0,T)$ is open, we won't be able to control continuous functions on it (since $(0,T)$ isn't compact.)

So consider $[1/m, T-1/m] \subset (0,T)$

let $w \in B_T$. then $|w| > 0$ on $[1/m, T-1/m]$

$\Rightarrow |w(t)| > 1/n$ for some n.

$\Rightarrow$ either $w(t) > 1/n$ for all $t \in [1/m, T-1/m]$

or $w(t) < -\frac{1}{n}$ for all $t \in [1/m, T-1/m]$.

Now we need to deal with the "open windows"
want to control w using countably many open windows (we've already seen it's impossible to do w/ finitely many.) $\Rightarrow$ consider the times in $[1/m, T-1/m]$ that occur at rational times.

i.e. $t \in [1/m, T-1/m]$ where $t = \frac{k}{N}$ for some $N \geq N_0$
$k \in \mathbb{Z}$ so that $\frac{k}{n} \in [1/m, T-1/m]$

(note if $[1/m, T-1/n]$ is short, we may not have times of the form $\frac{k}{3}$ in the interval. Need to choose the denominator big enough, hence the N_0)

<u>claim</u>:

$$B_T = \bigcap_{m=m_0}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{N=N_0}^{\infty} \{w \mid |w(k/N)| > \tfrac{1}{n} \text{ for all rational } \tfrac{k}{N} \text{ in } [1/m, T-1/m]\}.$$

<u>Notes</u>

$\{w \mid |w(k/N)| > \frac{1}{n}$ for all rational $\frac{k}{N}$ in $[1/m, T-1/m]\}$

is a set in $\mathcal{G}$ since there are only finitely many rational $\frac{k}{N}$ in $[1/m, T-1/m]$

once N, T, and m have been fixed.

So certainly I've written down a $G_{\delta\sigma\delta}$ set.

Now I need to prove $B_T = \cap \cup \cap$

($\Longrightarrow$) let $w \in B_T$. then we need that

"for every $m \geq m_0$ there is a $n \geq 1$ (depending on m) so that $|w(k/N)| > 1/n$ for all $N \geq N_0$ and all k for which $k/N \in [1/m, T-1/m]$."

this is true by the construction and the continuity of w.

Now assume

$\omega \in \bigcap \bigcup \bigcap$. I want to show $\omega \in B_T$.

It suffices to show $w(t) \neq 0$ $\forall t \in (0, T)$.

Assume not. Then $\exists$ t_0 so that $w(t_0) = 0$.

First of all $t_0 \in [1/m, T - 1/m]$ for some m.

And $\exists$ n so that

$$|w(k/N)| > 1/n \quad \text{for all}$$

rationals k/N in $[1/m, T - 1/m]$. But this is impossible because by continuity $\exists \delta$ so that

$$|s - t_0| < \delta \Rightarrow |w(s) - w(t_0)| = |w(s)| < \frac{1}{2n}$$

and since there are rationals of the form k/N in that δ-ball around t_0 we have a contradiction. Thus w cannot vanish on $(0, T)$ $\Rightarrow$ $w \in B_T$ as desired. //