Third Homework set.

Problem 1:

assume (X, τ) is T_4 and $F_1, F_2 \subset X$ are disjoint and closed. Prove $\exists\ f: X \to \mathbb{R}$ so that f is continuous (assume usual metric topology on $\mathbb{R}$), so that $0 \le f(x) \le 1 \quad \forall x \in X$ and so that $f|_{F_1} = 0$ and $f|_{F_2} = 1$

proof:

From class, if $\Delta = \left\{ \frac{k}{2^n} \;\middle|\; 0 < k < 2^{n-1} \quad k \in \mathbb{N}, n \in \mathbb{N} \right\}$ are the dyadic numbers in $(0,1)$, we know $\exists$ a family of open sets that satisfy:

$$F_1 \subset U_r \quad \forall r \in \Delta$$

$$F_2 \subset X - [U_r] \quad \forall r \in \Delta$$

$$r < s,\ r, s \in \Delta \Rightarrow [U_r] \subset U_s.$$

We use $\{U_r\}_{r \in \Delta}$ to define our function f. First, let $U_1 = X$ and redefine Δ to include 1. Let $f(x) = \inf_{r \in \Delta} \{ r \mid x \in U_r \}$.

Then $f|_{F_1} = 0$ and $f|_{F_2} = 1$ by construction. Also, $0 \le f(x) \le 1$ by construction. So it remains to show f is continuous. i.e. if $U \subset \mathbb{R}$ is open then $f^{-1}(U)$ is open. (note: if $U \cap [0,1] = \emptyset$, then $f^{-1}(U) = \emptyset$ which is open.) Let $V = f^{-1}(U)$ and $x \in V$. Then $\exists\ \varepsilon > 0$ so that $(f(x) - \varepsilon, f(x) + \varepsilon) \subset U$. Choose r_0 and $s_0 \in \Delta$ so that $f(x) - \varepsilon < r_0 < f(x) < s_0 < f(x) + \varepsilon$.

I claim that

$(X-[U_{r_0}]) \cap U_{s_0}$ is an open set that contains x and that is contained in V.

If $y \in U_{s_0}$ then $f(y) \le s_0 < f(x)+\varepsilon$.

if $y \in X-[U_{r_0}]$ then $f(y) \ge r_0 > f(x)-\varepsilon$.

$\Rightarrow f\left((X-[U_{r_0}]) \cap U_{s_0} \right) \subset U$. Also, $x \in$ the set

since $f(x) < s_0 \Rightarrow x \in U_{s_0}$ and $f(x) > r_0 \Rightarrow x \notin [U_{r_0}]$.

In this way, we've found an open set that contains x and is contained in $f^{-1}(U)$. $\Rightarrow f^{-1}(U)$ is open, as desired. //

<u>Problem 2:</u> (X,τ) is completely regular if given $F \subset X$, F closed and $x_0 \in X-F$ then $\exists$ continuous real-valued $f: X \to \mathbb{R}$ (usual metric topology on $\mathbb{R}$) so that

$0 \le f(x) \le 1 \ \forall x \in X$, $f(x_0) = 0$ and $f|_F = 1$.

<u>claim:</u> $T_4 \Rightarrow$ completely regular.

<u>proof:</u> since X is T_4, X is T_1 and thus $\{x_0\}$ is a closed set. By previous problem, $\exists$ function f with desired properties.

<u>claim:</u> if (X, τ) is completely regular and $A \subset X$ then (A, τ_A) is completely regular.

<u>proof:</u> Let $x_0 \in A$ and $F \subset A$ closed $x_0 \notin F$.
Since F is closed in A, $F = G \cap A$ for some G closed in X. Since (X, τ) is completely regular, $\exists$ $f : X \to \mathbb{R}$ so that $f(x_0) = 0$ and $f|_G = 1$ and f is continuous wrt τ and $f(x) \in [0,1]$ $\forall x \in X$.

Let $f|_A : A \to \mathbb{R}$. Then $f|_A(x_0) = 0$ and $f|_A(x) = 1$ $\forall$ $x \in F$. $f|_A(x) \in [0,1]$ $\forall x \in A$. It remains to prove
$f|_A$ is continuous. Let $U \subset \mathbb{R}$ be open
then $f^{-1}(U) \in \tau$. $\Rightarrow$ $f^{-1}(U) \cap A \in \tau_A$
$\Rightarrow$ $f|_A^{-1}(U) \in \tau_A$ $\Rightarrow$ $f|_A$ is continuous. //

<u>claim:</u> completely regular $\not\Rightarrow$ T4.

<u>note:</u> this must be true because a subspace of a completely regular space is also completely regular. T4 is not inherited by subspaces.

for a nice example of a completely regular space that isn't a T4 space, see example 14.6 of Kazuo Yokoyama's notes on our course webpage.

problem 3:

Let $U(\phi, m, R, \varepsilon) = \{\psi \in X \mid \max\limits_{0 \le k \le m} \sup\limits_{x \in [-R, R]} |\phi^{(k)}(x) - \psi^{(k)}(x)| < \varepsilon\}$

where $X = \{$ infinitely differentiable functions on $\mathbb{R}\}$.

define $N(\phi) = \{U(\phi, m, R, \varepsilon) \mid m \in \mathbb{N}, R > 0, \varepsilon > 0\}$.

a) verify $U(\phi, m', R', \varepsilon') \subset U(\phi, m, R, \varepsilon)$ if

$m' \ge m$, $R' \ge R$, $\varepsilon' \le \varepsilon$.

proof: let $\psi \in U(\phi, m', R', \varepsilon')$. then

$$\max_{0 \le k \le m'} \max_{-R' \le x \le R'} |\phi^{(k)} - \psi^{(k)}| < \varepsilon'$$

$$\Rightarrow \max_{0 \le k \le m} \max_{-R' \le x \le R'} |\phi^{(k)}(x) - \psi^{(k)}(x)| < \varepsilon'$$

$$\Rightarrow \max_{0 \le k \le m} \max_{-R \le x \le R} |\phi^{(k)}(x) - \psi^{(k)}(x)| < \varepsilon'$$

$$\Rightarrow \max_{0 \le k \le m} \max_{-R \le x \le R} |\phi^{(k)}(x) - \psi^{(k)}(x)| < \varepsilon$$

$\Rightarrow \psi \in U(\phi, m, R, \varepsilon)$. //

b) Verify $N(\phi)$ satisfies the properties of a local base at ϕ.

first, $\phi \in U(\phi, m, R, \varepsilon)$ $\forall\, m, R, \varepsilon$ so each member of $N(\phi)$ contains ϕ.

Secondly, the intersection of members of $N(\phi)$ contains a member of $N(\phi)$:

$U(\phi, m, R, \varepsilon) \cap U(\phi, m', R', \varepsilon')$

contains $U(\phi, \tilde{m}, \tilde{R}, \tilde{\varepsilon}) \in N(\phi)$

where $\tilde{m} = \max\{m, m'\}$, $\tilde{R} = \max\{R, R'\}$ and $\tilde{\varepsilon} = \min\{\varepsilon, \varepsilon'\}$.

c) (X, τ) is first countable since a countable local base at ϕ is

$\{U(\phi, m, n, 1/k) \mid m \in \mathbb{N}, n \in \mathbb{N}, 1/k\}$.

this follows since given $U(\phi, m, R, \varepsilon)$ we take $n \in \mathbb{N}$, $n \geq R$ and $k \in \mathbb{N}$, $1/k < \varepsilon$ then

$U(\phi, m, n, 1/k) \subset U(\phi, m, R, \varepsilon)$.

to show (X, τ) is second countable we need to find a countable dense set. I claim

$$A = \{\text{polynomials w/ rational coeffs}\}$$

is dense. A is certainly $\subset X$ since ~~they're~~ polynomials are infinitely differentiable and $A = \bigcup_0^\infty A_k$ where

$$A_k = \{\text{polynomials of degree } k, \text{ rational coeffs}\}$$

A_k is countable and the countable union of countable sets is countable. $\Rightarrow A$ is countable

It remains to show A is dense

Let $\phi \in X$ and $\phi \in U \in \tau$. Then $\exists$ m, R, ε $\ni$

$$U(\phi, m, R, \varepsilon) \subset U.$$

It suffices to find $\psi \in A$ so that $\psi \in U(\phi, m, R, \varepsilon)$.

By Stone-Weierstrauss theorem, given f continuous on $[-R, R]$ $\exists$ polynomial p so that

$\max\limits_{x \in [-R, R]} |f(x) - p(x)| < \varepsilon/2$. I claim there's a polynomial $\tilde{p} \in A$ so that $\max |f(x) - \tilde{p}| < \varepsilon$.

why? $p(x) = \sum_0^L c_j x^j$ for some $c_j \in \mathbb{R}$.

take $\tilde{c}_j \in \mathbb{Q}$ so that $|c_j - \tilde{c}_j| < \delta \quad 0 \le j \le L$

then
$$\max |p(x) - \tilde{p}(x)| = \max \left| \sum_0^L (c_j - \tilde{c}_j) x^j \right|$$
$$\le \max \sum_0^L |c_j - \tilde{c}_j| \, |x|^j$$
$$\le R^L (L+1) \delta \qquad \text{(here, I assumed } R \ge 1\text{)}$$
$$< \varepsilon/2 \quad \text{for } \delta \text{ sufficiently small}$$

$$\Rightarrow \max |f(x) - \tilde{p}(x)| \le \max |f(x) - p(x)| + \max |p(x) - \tilde{p}(x)|$$
$$< \varepsilon.$$

$\Rightarrow$ polynomials w/ rational coeffs are dense in $L^\infty([-R, R])$ (continuous functions on $[-R, R]$ with L^∞ metric)

$\Rightarrow \exists\, g_m \in A$ so that

$$\max_{[-R,R]} |f^{(m)}(x) - g_m(x)| < \delta \qquad (\delta \text{ to be chosen later.})$$

we integrate g_m to get another member of A, call it g_{m-1}. (Note! Need to choose integration constant in $\mathbb{Q}$!) Then

$$f^{(m-1)}(x) - g_{m-1}(x) = f^{(m-1)}(-R) - g_{m-1}(-R) + \int_{-R}^{x} f^{(m)}(y) - g^{(m)}(y)\, dy$$

$$\Rightarrow \left|f^{(m-1)}(x) - g_{m-1}(x)\right|$$
$$\leq \left|f^{(m-1)}(-R) - g_{m-1}(-R)\right| + 2R\delta$$

choose the integration constant $g_{m-1}(-R)$ so that $g_{m-1}(-R) \in \mathbb{Q}$ and

$$\left|f^{(m-1)}(-R) - g_{m-1}(-R)\right| < R\delta.$$

$$\Rightarrow \max_{x \in [-R,R]} \left|f^{(m-1)}(x) - g_{m-1}(x)\right| < 3R\delta$$

proceeding by induction,

$$\max_{x \in [-R,R]} \left|f^{(m-k)}(x) - g_{m-k}(x)\right| < (3R)^k \delta$$

$$0 \leq k \leq m$$

$$\Rightarrow \max_{0 \leq k \leq m} \max_{x \in [-R,R]} \left|f^{(k)}(x) - G^{(k)}(x)\right| < (3R)^m \delta \quad \binom{\text{since}}{R \geq 1}$$

where G is defined to be g_{m-m} the result of integrating our original g_m m times.

$\Rightarrow$ if we choose $\delta < \varepsilon/(3R)^m$ then we're done we've found our $G \in U(\phi, m, R, \varepsilon)$, as desired //

d) can we define

$$\rho(\phi, \psi) = \sup_{k \geq 0} \sup_{x \in \mathbb{R}} |\phi^{(k)}(x) - \psi^{(k)}(x)|$$

and have it be a metric ~~for~~ on X?

No... it's not finite if $\phi(x) = x^2$ and $\psi(x) = x^3$ then $\rho(\phi, \psi) = \infty$.

e) show $f : X \to \mathbb{R}$ defined by

$$f(\phi) = \int_{-5}^{8} \left(\phi^{(2)}(x)\right)^3 dx$$

is continuous. (usual metric on $\mathbb{R}$)

proof: let $U \subset \mathbb{R}$ be open. Take $y_0 \in U$. then $\exists\ \varepsilon > 0 \ni (y_0 - \varepsilon, y_0 + \varepsilon) \subset U$.

Consider $f^{-1}(U)$. Let $\phi \in f^{-1}(U)$.

[note: $f^{-1}(U) \neq \emptyset$ since we can take

$\phi(x) = \sqrt[3]{y_0/11}\, x^2 \in X$ and $f(\phi) = y$. But it doesn't really matter, since if $f^{-1}(U) = \emptyset$ then $f^{-1}(U)$ is open since $\emptyset$ is open.]

I'll show that if I take δ small enough then $U(\phi, 2, 8, \delta) \subset f^{-1}(U)$

and $\Rightarrow f^{-1}(U)$ is open.

$$|F(\phi) - F(\psi)| = \left| \int_{-3}^{8} (\phi^{[2]}(x))^3 - (\psi''(x))^3 \, dx \right|$$

$$\leq \int_{-8}^{8} |(\phi''(x))^3 - (\psi''(x))^3| \, dx$$

$$\leq \int_{-8}^{8} |\phi''(x) - \psi''(x)| \left\{ |\phi''(x)|^2 + |\phi''(x)||\psi''(x)| + |\psi''(x)|^2 \right\} dx$$

$$\leq 3M^2 \cdot 16 \cdot \delta$$

Where M is a uniform upper bound of ϕ'' and ψ'' if $\psi \in U(\phi, 2, 8, \delta)$

$M = \max|\phi''| + \delta$ works, for example

then by taking δ sufficiently small, we get $3M^2 \cdot 16\delta < \varepsilon \Rightarrow U(\phi, 2, 8, \delta) \subset f^{-1}(U)$ as desired. //

problem 4: Consider

$$\frac{dx(t)}{dt} = y$$

$$\frac{dy}{dt} = x \qquad x, y \in \mathbb{R}.$$

given initial data $x(0) = a$ and $y(0) = b$

we have exact solution

$$x(t) = a\cosh(t) + b\sinh(t)$$

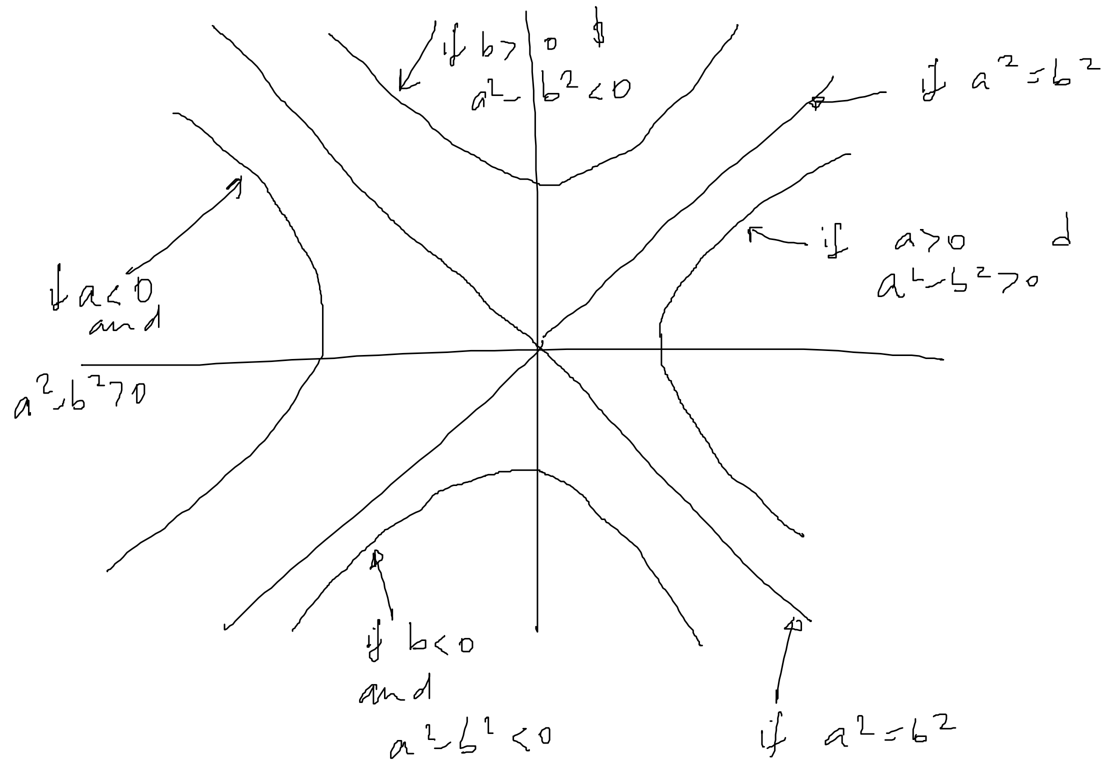
$$y(t) = a\sinh(t) + b\cosh(t).$$

these solutions satisfy

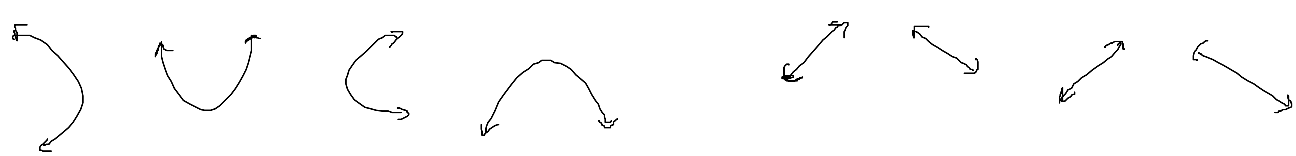
$$x(t)^2 - y(t)^2 = a^2 - b^2$$

$\Rightarrow$ $(x(t), y(t))$ lies in the level set

$$x^2 - y^2 = a^2 - b^2$$



So given initial data (a,b) the resulting orbit is one of the following:



or $\cdot$ $(0,0)$

define a topology on $X = \{\text{set of orbits}\}$
by $U \subset X$ is open if
$V = \{ (a,b) \mid \{(a\cosh(t)+b\sinh(t), a\sinh(t)+b\cosh(t)) \mid t \in \mathbb{R}\} \subset U \}$

is open in $\mathbb{R}^2$.

Prove this is a topology.

1) X is open? yes since $\mathbb{R}^2$ is open and the orbits w/ initial data in $\mathbb{R}^2$ will give all of X.

2) $\emptyset$ is open? yes, since $\emptyset$ is open in $\mathbb{R}^2$

3) $\bigcup_\alpha U_\alpha$ is open?

fix α then

$\left\{ \begin{array}{l}\text{initial conditions that yield}\\ \text{the orbits in } U_\alpha\end{array} \right\} = V_\alpha$

and V_α is open in $\mathbb{R}^2$

$\Rightarrow \bigcup_\alpha V_\alpha$ is open in $\mathbb{R}^2 \Rightarrow \bigcup_\alpha U_\alpha$ is open

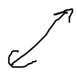
4) $U_\alpha \cap U_\beta$ is open? same argument as above, we know $V_\alpha \cap V_\beta$ is open in $\mathbb{R}^2$.

Note: the above works for any topology on $\mathbb{R}^2$.

<u>claim</u>: if we consider $\mathbb{R}^2$ with usual metric topology, then X with the induced topology is not Hausdorff.

<u>proof</u>: consider the following two orbits:

$$x_0 = (0,0) \in X$$

$x_1 =$  $\in X$.

then x_0 arises from the initial data $(0,0)$, and x_1 arises from the initial data (a,a), $a>0$.

If $x_0 \in U$ and $x_1 \in \tilde{U}$ with $U \cap \tilde{U} = \emptyset$ and U and $\tilde{U}$ open, then we have found open sets V and $\tilde{V}$ in $\mathbb{R}^2$ so that $(0,0) \in V$ and $\{(a,a) \mid a>0\} \subset \tilde{V}$ and $V \cap \tilde{V} = \emptyset$. this is impossible since $(0,0)$ is a limit point* of $\{(a,a) \mid a>0\}$

$\Rightarrow (X, \tau)$ is not Hausdorff. //

* here's where we use that the topology on $\mathbb{R}^2$ is the usual metric topology.

problem 5. M = real invertible matrices

$m \sim g$ if $\exists\ p \in M$ so that

$$m = pgp^{-1}$$

i.e. m is similar to g.

We define the orbit that contains m to be $\{g \in M \mid m = pgp^{-1} \text{ some } p \in M\}$.

Let X be the set of orbits.

a) If we're looking at 2×2 matrices then the jordan canonical form theorem tells us that a matrix

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is similar to one of

the following: $\underset{b \neq a}{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}}$, $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$

since M = invertible, we know $a \neq 0, b \neq 0$

$\Rightarrow$ the orbits are

$$\left\{ \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \, a, b \in \mathbb{R} - \{0\} \right\} \cup \left\{ \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \middle| \, a \in \mathbb{R} - \{0\} \right\} \right\}$$

b) we endow X with the quotient topology, as before. We view M a subset of $\mathbb{R}^{n^2}$ with the ℓ^2 metric on $\mathbb{R}^{n^2}$.

X is not hausdorff

consider the $\mathbb{R}^2$ case.

let x_0 be the orbit that contains $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and let x_ε be the orbit that contains $\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$

the orbit that contains $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is like the fixed point $(0,0)$ in the previous problem

i.e. $p\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}p^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\forall$ invertible p.

On the other hand if $p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$p\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}p^{-1} = \begin{pmatrix} 1 - \frac{\varepsilon ac}{ad-bc} & \frac{\varepsilon a^2}{ad-bc} \\ \frac{-\varepsilon c^2}{ad-bc} & 1 + \frac{\varepsilon ac}{ad-bc} \end{pmatrix}$$

$\Rightarrow$ every matrix of that form will end up in the equivalence class of $x_\varepsilon = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$.

$\Rightarrow$ orbit containing $\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$ arises from the matrices in M

$$V_\varepsilon = \left\{ \begin{pmatrix} 1 - \frac{\varepsilon ac}{ad-bc} & \frac{\varepsilon a^2}{ad-bc} \\ -\frac{\varepsilon c^2}{ad-bc} & 1 + \frac{\varepsilon ac}{ad-bc} \end{pmatrix} \,\middle|\, \begin{matrix} a,b,c,d \in \mathbb{R} \\ ad-bc \neq 0 \end{matrix} \right\}$$

(In the previous problem, we were varying the time parameter t and as $t \downarrow -\infty$ the point $(x(t), y(t))$ had the limiting value $(0,0)$. This stopped the quotient space from being Hausdorff. Here, we'll vary a, b, c, d in a cunning way.)

$$\rho\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 - \frac{\varepsilon ac}{det} & \frac{\varepsilon a^2}{det} \\ -\varepsilon c^2/det & 1 + \varepsilon ac/det \end{pmatrix}\right) = \frac{\varepsilon (a^2 + c^2)}{ad - bc}$$

so if we take $a, c \downarrow 0$ while taking $b, d \uparrow \infty$ in such a way that $ad - bc = 1$ then we've shown that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a limit point of the set

$$\left\{ \begin{pmatrix} 1 - \frac{\varepsilon ac}{det} & \frac{\varepsilon a^2}{det} \\ -\varepsilon c^2/det & 1 + \varepsilon ac/det \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$$

$\Rightarrow$ we cannot separate $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ from the set V_ε in $\mathbb{R}^{2^2}$

$\Rightarrow$ we cannot separate the orbit arising from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ from the orbit that arises from $\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$. $\Rightarrow (X, \tau)$ is not Hausdorff.

The case of $n \times n$ matrices is done similarly – just put one ε in above the diagonal