HW assignment 2.

1) Show the Sobolev metric is a metric. The symmetry and non-degeneracy are immediate. We just need to show the triangle inequality.

Note $\rho(f,g) = \rho(f-g, 0)$ and so all we really need to prove is
$$\rho(f+g, 0) \leq \rho(f,0) + \rho(g,0).$$

Let $I = \int \sum_0^k |f^{(i)}(x) - g^{(i)}(x)|^p dx = (\rho(f+g,0))^p$

then
$$I = \int \sum_0^k |f^{(i)} - g^{(i)}| \, |f^{(i)} - g^{(i)}|^{p-1} dx$$

$$\leq \int \sum_0^k |f^{(i)}| \, |f^{(i)} - g^{(i)}|^{p-1} dx + \int \sum_0^k |g^{(i)}| \, |f^{(i)} - g^{(i)}|^{p-1} dx$$

$$\leq \int \sqrt[p]{\sum_0^k |f^{(i)}|^p} \sqrt[q]{\sum_0^k |f^{(i)} - g^{(i)}|^p} \, dx + \int \sqrt[p]{\sum_0^k |g^{(i)}|^p} \sqrt[q]{\sum_0^k |f^{(i)} - g^{(i)}|^p} \, dx$$ Hölder for sums

$$\leq \sqrt[p]{\int \sum_0^k |f^{(i)}|^p dx} \sqrt[q]{\int \sum_0^k |f^{(i)} - g^{(i)}|^p dx} + \sqrt[p]{\int \sum_0^k |g^{(i)}|^p dx} \sqrt[q]{\int \sum_0^k |f^{(i)} - g^{(i)}|^p dx}$$ Hölder integral ineq

$$\Rightarrow I \leq \rho(f,0) I^{1/q} + \rho(g,0) I^{1/q} \Rightarrow \rho(f+g,0) \leq \rho(f,0) + \rho(g,0).$$

Okay, we just proved the triangle inequality for $1 < p < \infty$. We now have to prove it for $p = 1$ and $p = \infty$.

$p=1$:

$$\rho_{1,k}(f+g,0) = \int \sum_{0}^{k} |f^{(i)}(x) + g^{(i)}(x)| \, dx$$

$$\leq \int \sum_{0}^{k} |f^{(i)}(x)| + |g^{(i)}(x)| \, dx$$

$$= \int \sum_{0}^{k} |f^{(i)}(x)| \, dx + \int \sum_{0}^{k} |g^{(i)}(x)| \, dx$$

$$= \rho_{1,k}(f,0) + \rho_{1,k}(g,0)$$

$p = \infty$:

$$\rho_{\infty,k}(f+g,0) = \sup_{x \in [a,b]} |f^{(i)}(x) + g^{(i)}(x)|$$

or max since they're continuous and $[a,b]$ is closed + compact

$$\leq \sup |f^{(i)}(x)| + |g^{(i)}(x)|$$

$$\leq \sup |f^{(i)}(x)| + \sup |g^{(i)}(x)|$$

$$= \rho_{\infty,k}(f,0) + \rho_{\infty,k}(g,0).$$

let $$f_n(x) = \frac{1}{n^{7/2}} \cos\left(2\pi n \frac{x-a}{b-a}\right)$$

$$f_n'(x) = -\frac{1}{n^{5/2}} \frac{2\pi}{b-a} \sin\left(n 2\pi \frac{x-a}{b-a}\right)$$

$$f_n^{(2)}(x) = \frac{1}{n^{3/2}} \left(\frac{2\pi}{b-a}\right)^2 \cos\left(n 2\pi \frac{x-a}{b-a}\right)$$

$$f_n^{(3)}(x) = \frac{-1}{\sqrt{n}} \left(\frac{2\pi}{b-a}\right)^3 \sin\left(n\pi \frac{x-a}{b-a}\right)$$

$$f_n^{(4)}(x) = \sqrt{n} \left(\frac{2\pi}{b-a}\right)^4 \cos\left(n\pi \frac{x-a}{b-a}\right)$$

$$\rho_{2,3}(f_n, 0) = \sqrt{\int_a^b \frac{1}{n^7}|\cos|^2 + \frac{1}{n^5}\left(\frac{2\pi}{b-a}\right)^2 |\sin|^2 + \frac{1}{n^3}\left(\frac{2\pi}{b-a}\right)^4 |\cos|^2 + \frac{1}{n}\left(\frac{2\pi}{b-a}\right)^4 |\sin|^2 \, dx}$$

$$\leq \sqrt{\int_a^b 4M \frac{1}{n}\, dx} = \frac{1}{\sqrt{n}} \sqrt{4M(b-a)}$$

where $M = \max\left\{1, \left(\frac{2\pi}{b-a}\right)^2, \left(\frac{2\pi}{b-a}\right)^4, \left(\frac{2\pi}{b-a}\right)^6\right\}$

$\Rightarrow \rho_{2,3}(f_n, 0) \to 0$ as $n \to \infty$

On the other hand,

$$\rho_{2,4}(f_n, 0) \geq \sqrt{\int_a^b \left|f_n^{(4)}(x)\right|^2 dx}$$

$$= \sqrt{\int_a^b n\left(\frac{2\pi}{b-a}\right)^8 \cos\left(n 2\pi \frac{x-a}{b-a}\right)^2 dx}$$

$$= \sqrt{n}\left(\frac{2\pi}{b-a}\right)^4 \sqrt{\int_0^{n\cdot 2\pi} \cos^2(y)\, dy \cdot \frac{b-a}{n\cdot 2\pi}}$$

$$= \sqrt{n}\left(\frac{2\pi}{b-a}\right)^4 \sqrt{n\cdot\pi\cdot\frac{b-a}{n\cdot 2\pi}}$$

$$= \sqrt{n}\left(\frac{2\pi}{b-a}\right)^4 \sqrt{\frac{b-a}{2}}$$

So $\rho_{2,4}(f_n, 0)$ has a lower bound which goes to ∞ as $n \to \infty$. $\Rightarrow$ $\rho_{2,4}(f_n, 0)$ diverges even though $\rho_{2,3}(f_n, 0) \to 0$.

Problem #2

By the same logic as before, for the triangle inequality, we only have to check

$$\rho((f_1+f_2, g_1+g_2), (0,0)) \le \rho((f_1,g_1),(0,0)) + \rho((f_2,g_2),(0,0))$$

$$\rho((f_1+f_2, g_1+g_2), (0,0))^2$$

$$= \frac{1}{2}\int (f_1'+f_2')^2 + (g_1+g_2)^2 \, dx$$

$$\le \frac{1}{2}\int |f_1'||f_1'+f_2'| + |g_1||g_1+g_2| + |f_2'||f_1'+f_2'| + |g_2||g_1+g_2| \quad dx$$

$$\le \frac{1}{2}\int \sqrt{|f_1'|^2+|g_1|^2}\sqrt{|f_1'+f_2'|^2+|g_1+g_2|^2} + \sqrt{|f_2'|^2+|g_2|^2}\sqrt{|f_1'+f_2'|^2+|g_1+g_2|^2} \, dx$$

Hölder for sums

$$\le \frac{1}{2}\sqrt{\int |f_1'|^2+|g_1|^2 dx}\sqrt{\int |f_1'+f_2'|^2+|g_1+g_2|^2 dx} + \frac{1}{2}\sqrt{\int |f_2'|^2+|g_2|^2 dx}\sqrt{\int |f_1'+f_2'|^2+|g_1+g_2|^2 dx}$$

$$\Rightarrow \frac{1}{2}\sqrt{\int |f_1'+f_2'|^2+|g_1+g_2|^2 dx} \le \frac{1}{2}\sqrt{\int |f_1'|^2+|g_1|^2 dx} + \frac{1}{2}\sqrt{\int |f_2'|^2+|g_2|^2 dx}$$

$$\Rightarrow \rho((f_1+f_2, g_1+g_2), (0,0)) \le \rho((f_1,g_1),(0,0)) + \rho((f_2,g_2),(0,0))$$

this proves the triangle inequality. Symmetry is immediate. Now just need to check non degeneracy:

$$\rho((f,g),(0,0)) = 0 \overset{?}{\underset{?}{\Rightarrow}} f \equiv 0 \text{ and } g \equiv 0$$

if $\int_0^1 f_x(x)^2 + g(x)^2 \, dx = 0$ and f_x and g are continuous, then $f_x \equiv 0$ and $g \equiv 0$. This implies $f(x) \equiv$ constant. But since $(f,g) \in X$ we know $f(0) = f(1) = 0 \Rightarrow f(x) \equiv 0$ this shows $(f,g) = (0,0)$ as desired.

2b claim:

$$\rho((q(t,0), q_t(t,0)), (0,0)) = \rho((f,g),(0,0))$$

Let $F(t) = \int_0^1 \frac{\partial q}{\partial x}(t,x)^2 + \frac{\partial q}{\partial t}(t,x)^2 \, dx$

$$= 2\rho((q(t,0), q_t(t,0)),(0,0))^2$$

then
$$\frac{dF}{dt} = 2\int_0^1 \frac{\partial q}{\partial x}\frac{\partial^2 q}{\partial x \partial t} + \frac{\partial q}{\partial t}\frac{\partial^2 q}{\partial t^2} \, dx$$

$$= 2\int_0^1 \frac{\partial q}{\partial x}\frac{\partial^2 q}{\partial x \partial t} + \frac{\partial q}{\partial t}\frac{\partial^2 q}{\partial x^2} \, dx \quad \text{because } q_{tt} = q_{xx}$$

$$= -2\int_0^1 \frac{\partial^2 q}{\partial x^2}\frac{\partial q}{\partial t} \, dx + 2\left.\frac{\partial q}{\partial x}\frac{\partial q}{\partial t}\right|_0^1 + 2\int_0^1 \frac{\partial q}{\partial t}\frac{\partial^2 q}{\partial x^2} \, dx$$

$$\Rightarrow \frac{dF}{dt} = 2\frac{\partial q}{\partial x}(t,1)\frac{\partial q}{\partial t}(t,1) - 2\frac{\partial q}{\partial x}(t,0)\frac{\partial q}{\partial t}(t,0)$$

but $q(t,0) = 0 \quad \forall t \quad \Rightarrow \frac{\partial q}{\partial t}(t,0) = 0$

$ q(t,1) = 0 \quad \forall t \quad \Rightarrow \frac{\partial q}{\partial t}(t,1) = 0$

$\Rightarrow \frac{dF}{dt} = 0$ for all times.

$\Rightarrow F(t) = F(0)$

$\Rightarrow \rho((q(t,\cdot), q_t(t,\cdot)), (0,0)) = \rho((f,g),(0,0))$.

c) Assume we have 2 solutions q_1 and q_2 w/ the same initial conditions. Then their difference $W = q_1 - q_2$ satisfies

$$W_{tt} = W_{xx}$$
$$W(0,x) = 0 \quad \forall x$$
$$W_t(0,x) = 0 \quad \forall x$$
$$W(t,0) = W(t,1) = 0 \quad \forall t$$

$\Rightarrow \rho((W(t,\cdot), W_t(t,\cdot)), (0,0)) = \rho((0,0),(0,0)) = 0$

$\Rightarrow \begin{matrix} W(t,\cdot) \equiv 0 \\ W_t(t,\cdot) \equiv 0 \end{matrix} \quad \forall t \quad \Rightarrow W(t,x) = 0 \quad \forall t, \forall x.$

$\Rightarrow q_1 = q_2$ as desired.

Problem 3: Here we have

$$u_{tt}(t,\vec{x}) = \nabla\cdot A\nabla u \qquad \text{where } A_{ij} = a_{ij}$$

$$u(0,\vec{x}) = f(\vec{x}) \qquad \forall \vec{x}\in D$$

$$u_t(0,\vec{x}) = g(\vec{x}) \qquad \forall \vec{x}\in D$$

$$u(t,\vec{x}) = 0 \qquad \forall t \ \forall x \in \partial D$$

define the metric on X by

$$\rho((f,g),(F,G)) = \sqrt{\frac{1}{2}\int_D (A\nabla(f-F))\cdot\nabla(f-F) + (g-G)^2\,dx}$$

then

$$\rho((u(t,\cdot),u_t(t,\cdot)),(0,0))^2$$

$$= \frac{1}{2}\int_D (A\nabla u)\cdot\nabla u + u_t^2\,dx \quad =: F(t)$$

$$\frac{dF}{dt} = \frac{1}{2}\int_D (A\nabla u_t)\cdot\nabla u + (A\nabla u)\cdot\nabla u_t + 2u_tu_{tt}\,dx$$

$$= \frac{1}{2}\int_D (\nabla u_t)\cdot(A^T\nabla u) + (A\nabla u)\cdot\nabla u_t + 2u_tu_{tt}\,dx$$

since $A = A^T$

$$= \int_D (A\nabla u)\cdot\nabla u_t + u_tu_{tt}\,dx$$

$$= \int_D \nabla\cdot(u_tA\nabla u) - u_t\nabla\cdot(A\nabla u) + u_tu_{tt}\,dx$$

since $u_{tt} = \nabla \cdot (A \nabla u)$,
the 2nd and 3rd terms cancel.

$$\Rightarrow \frac{dF}{dt} = \int_D \nabla \cdot (u_t A \nabla u) \, dx$$

$$= \int_{\partial D} u_t A \nabla u \cdot \vec{n} \, ds \quad \text{(divergence theorem)}$$

$= 0$ because $u(t, x) = 0 \;\; \forall x \in \partial D$
$\Rightarrow u_t(t, x) = 0 \;\; \forall x \in \partial D$

this shows that at all times,
the solution is
a constant distance from $(0, 0)$.

By the same argument as before, this implies that solutions are unique, because if there are two solutions w/ the same initial data then their difference is a solution to the anisotropic wave equation with $(0, 0)$ as initial data.

It remains to show that ρ is actually a metric.

First, from linear algebra, because A is symmetric,

$$A = BDB^{-1}$$ for some orthogonal matrix B and some diagonal matrix D.

A is positive definite $\Rightarrow$ $A\vec{v} \cdot \vec{v} > 0 \quad \forall \vec{v} \neq 0$

this means that all of the entries on the diagonal of D are positive

$$\Rightarrow \quad A = \sqrt{A}\sqrt{A} \quad \text{where}$$

$$\sqrt{A} := B\sqrt{D}B^{-1} \quad \text{and}$$

$\sqrt{D}$ is the matrix with $(\sqrt{D})_{ij} = \begin{cases} 0 & i \neq j \\ \sqrt{D_{ii}} & i = j \end{cases}$

thus

$$A\vec{v} \cdot \vec{w} = (\sqrt{A}\vec{v}) \cdot (\sqrt{A}\vec{w})$$ for all vectors $\vec{v}$ and $\vec{w}$

and $$A\vec{v} \cdot \vec{v} = (\sqrt{A}\vec{v}) \cdot (\sqrt{A}\vec{v}) = |\sqrt{A}\vec{v}|^2$$

For the usual reasons, to prove the triangle inequality it suffices to show

$$\rho\big((f_1+f_2,\, g_1+g_2),(0,0)\big) \le \rho\big((f_1,g_1),(0,0)\big) + \rho\big((f_2,g_2),(0,0)\big)$$

where

$$\rho\big((f_1,g_1),(0,0)\big) = \sqrt{\frac{1}{2}\int_D |\sqrt{A}\,\nabla f_1|^2 + |g_1|^2\,dx}$$

Since $|\sqrt{A}\,\nabla f_1|^2 = (A\nabla f_1)\cdot\nabla f_1$

this follows from

$$\int |\sqrt{A}\,\nabla(f_1+f_2)|^2 + |g_1+g_2|^2\,dx$$

$$\le \int |\sqrt{A}\,\nabla f_1 + \sqrt{A}\,\nabla f_2|\,|\sqrt{A}\,\nabla(f_1+f_2)| + (|g_1|+|g_2|)\,|g_1+g_2|$$

$$\le \int |\sqrt{A}\,\nabla f_1|\,|\sqrt{A}\,\nabla(f_1+f_2)| + |\sqrt{A}\,\nabla f_2|\,|\sqrt{A}\,\nabla(f_1+f_2)| + |g_1|\,|g_1+g_2| + |g_2|\,|g_1+g_2| \quad dx$$

$$\le \int \sqrt{|\sqrt{A}\,\nabla f_1|^2 + |g_1|^2}\,\sqrt{|\sqrt{A}\,\nabla(f_1+f_2)|^2 + |g_1+g_2|^2}\,dx + \sqrt{|\sqrt{A}\,\nabla f_2|^2 + |g_2|^2}\,\sqrt{|\sqrt{A}\,\nabla(f_1+f_2)|^2 + |g_1+g_2|^2}\,dx$$

(Hölder's ineq. for sums)

$$\le \sqrt{\int |\sqrt{A}\,\nabla f_1|^2 + g_1^2\,dx}\,\sqrt{I} + \sqrt{\int |\sqrt{A}\,\nabla f_2|^2 + g_2^2\,dx}\,\sqrt{I}$$

$\Rightarrow \sqrt{A} \le \sqrt{\quad} + \sqrt{\quad}$ as desired.

This proves the triangle inequality. We now have to think about symmetry and non-degeneracy.

The symmetry is immediate.

To show nondegeneracy, we need to show

$$\rho((f,g),(0,0)) = 0 \Rightarrow f=0 \text{ and } g=0$$

if $\rho((f,g),(0,0)) = 0$ then we know

$$\int_D |\sqrt{A}\nabla f|^2 + g^2\,dx = 0$$

$\Rightarrow \sqrt{A}\nabla f = \vec{0}$ and $g=0$ because they're continuous functions of x.

if $\sqrt{A}\,\nabla f = \vec{0}$, we know $\nabla f = \vec{0}$. This is because A is positive definite.

$\Rightarrow f =$ constant function.

but $f = 0$ on $\partial\Omega$ since $(f,g) \in X$.

$\Rightarrow f \equiv 0 \quad \Rightarrow (f,g) = (0,0)$ as desired

This proves $\rho(\cdot,\cdot)$ is a metric.

Problem 4.

When is

$A : f \to Af$ a contraction?

$$\psi(x) = Af(x) = \phi(x) + \lambda \int_a^b k(x,y) f(y)\, dy$$

$$\left(\|A\phi - A\tilde{\phi}\|_p\right)^p = \int_a^b \left| \lambda \int_a^b k(x,y) \cancel{f(y)} \left(\phi(y) - \tilde{\phi}(y)\right) dy \right|^p dx$$

I want to show $\|A\phi - A\tilde{\phi}\|_p \le \alpha \|\phi - \tilde{\phi}\|_p$ some $\alpha < 1$

So it suffices to show $\left(\|A\phi - A\tilde{\phi}\|_p\right)^p \le \alpha^p \left(\|\phi - \tilde{\phi}\|_p\right)^p$ ..

$$\le |\lambda|^p \int_a^b \left| \int_a^b k(x,y) \left(\phi(y) - \tilde{\phi}(y)\right) dy \right|^p dx$$

$$\le |\lambda|^p \int_a^b \left(\int_a^b |k(x,y)|\, |\phi(y) - \tilde{\phi}(y)|\, dy \right)^p dx$$

$$\le |\lambda|^p \int_a^b \left[\int_a^b |k(x,y)|^q \, dy \right]^{p/q} \left[\int_a^b |\phi(y) - \tilde{\phi}(y)|^p dy \right] dx$$

$$= |\lambda|^p \int_a^b |\phi(y) - \tilde{\phi}(y)|^p dy \int_a^b \left[\int_a^b |k(x,y)|^q dy \right]^{p/q} dx$$

$$= \alpha^p \left(\|\phi - \tilde{\phi}\|_p\right)^p$$

where $\alpha^p = |\lambda|^p \int_a^b \left(\int |k(x,y)|^q dy \right)^{p/q} dx$

So if k is "small enough" or λ is "small enough" ...

the previous argument was for $1<p<\infty$. For $p=1$ we get

$$\|A\phi - A\tilde{\phi}\|_1 \leq |\lambda| \sup_{y\in[a,b]} \int_a^b |K(x,y)|\,dx \;\; \|\phi-\tilde{\phi}\|_1$$

So we need

$$|\lambda| \sup_{y\in[a,b]} \int_a^b |K(x,y)|\,dx \;<\; 1.$$

<u>problem 5:</u>

Jacobi Method $\quad \sum_j A_{ij} x_j = b_i \quad$ holds for the solution

$$A_{ii}\tilde{x}_i = b_i - \sum_{j\neq i} A_{ij} x_j$$

$$\to \tilde{x}_i = \frac{b_i}{A_{ii}} - \sum_{j\neq i} \frac{A_{ij}}{A_{ii}} x_j$$

so our mapping takes $\vec{x}$ and returns $\vec{y}$ where

$$y_i = \frac{b_i}{A_{ii}} - \sum_{j\neq i} \frac{A_{ij}}{A_{ii}} x_j$$

we want to show that

$$\|A\vec{x}_0 - A\vec{x}_1\| \leq \alpha \|\vec{x}_0 - \vec{x}_1\| \quad \text{for each } \vec{x}_0, \vec{x}_1 \in \mathbb{R}^n$$

I'll choose a convenient norm on $\mathbb{R}^n$

$$\|\vec{x}\| = \max_i |x_i|$$

so I'll show a contraction wrt this norm

$$\|A\vec{x}_0 - A\vec{x}_1\| = \max_i |(Ax_0)_i - (Ax_1)_i|$$

$$= \max_i \left| \sum_{j \neq i} \frac{A_{ij}}{A_{ii}} (x_0)_j - (x_1)_j \right|$$

$$\leq \max_i \sum_{j \neq i} \left|\frac{A_{ij}}{A_{ii}}\right| |(x_0)_j - (x_1)_j|$$

$$\leq \|\vec{x}_0 - \vec{x}_1\| \max_i \sum_{j \neq i} \left|\frac{A_{ij}}{A_{ii}}\right|$$

since $|(x_0)_j - (x_1)_j| \leq \|\vec{x}_0 - \vec{x}_1\|$ for each j

so I have a contraction if my matrix A satisfies

$$\max_i \sum_{j \neq i} \frac{|A_{ij}|}{|A_{ii}|} < 1$$

$\Rightarrow$ for each i $\quad \sum_{j \neq i} |A_{ij}| < |A_{ii}|$

i.e. if the matrix is diagonally dominant.

Note: It's possible that there are matrices that fail the

$$\max_i \sum_{j \neq i} \frac{|A_{ij}|}{|A_{ii}|} < 1$$

test but for which the Jacobi method still converges. My condition is necessary but not sufficient. The good thing is that my condition is easily checked by any computer that'd be used to do the Jacobi iteration.

Gauss-Seidel Iteration

given $\vec{x}$ we create $\tilde{x}$ so that the fixed point of $A : x \to \tilde{x}$ satisfies the problem $Ax = \vec{b}$.

$$\tilde{x}_i = \frac{b_i}{A_{ii}} - \sum_{j=1}^{i-1} \frac{A_{ij}}{A_{ii}} \tilde{x}_j - \sum_{i+1}^{n} \frac{A_{ij}}{A_{ii}} x_j$$

so to create $\tilde{x}_i$ I use the entries $\tilde{x}_1, \ldots \tilde{x}_{i-1}$ and $x_{i+1} \cdots x_n$.

Again, I'll use the l^∞ metric on $\mathbb{R}^n$.

I'll show that the iteration converges by showing that the errors decrease

Let x^{∞} be the solution of $Ax^{\infty} = b$

Define $\vec{e}^k = \vec{x}^k - \vec{x}^{\infty}$ the k^{th} error.

and $\vec{x}^1 = A\vec{x}^0 \quad \vec{x}^2 = A\vec{x}^1 \quad \ldots \quad \vec{x}^{k+1} = A\vec{x}^k$

Plugging $\vec{x}^{k+1}$ and $\vec{x}^k$ in, I find

$$e_i^{k+1} = -\sum_{1}^{i-1} \frac{A_{ij}}{A_{ii}} e_j^{k+1} - \sum_{i+1}^{n} \frac{A_{ij}}{A_{ii}} e_j^k$$

let $r_i = \sum_{j \neq i} \left| \frac{A_{ij}}{A_{ii}} \right|$

I claim that if $r = \max r_i < 1$ then the iteration converges.

Start at the first index

$$|e_1^{k+1}| \leq \sum_{2}^{n} \left| \frac{A_{1j}}{A_{11}} \right| |e_j^k| \leq \|e^k\| \sum_{2}^{n} \left| \frac{A_{1j}}{A_{11}} \right|$$

$$= \|e^k\| r_1 \leq r \|e^k\|$$

now the second coordinate

$$|e_2^{k+1}| \leq \left| \frac{A_{21}}{A_{22}} \right| |e_1^{k+1}| + \sum_{3}^{n} \left| \frac{A_{2j}}{A_{22}} \right| |e_j^k|$$

$\Rightarrow$

$$|e_2^{k+1}| \le \left|\frac{A_{21}}{A_{22}}\right| r \|\vec{e}^k\| + \sum_3^n \left|\frac{A_{2j}}{A_{22}}\right| |e_j^k|$$

because we've already controlled the first component of e^{k+1} using $r\|e^k\|$

$$\le \left|\frac{A_{21}}{A_{22}}\right| \|\vec{e}^k\| + \|\vec{e}^k\| \sum_3^n \left|\frac{A_{2j}}{A_{22}}\right|$$

because $r < 1$

$$= \|\vec{e}^k\| \sum_{j \neq 2} \left|\frac{A_{2j}}{A_{22}}\right| = \|\vec{e}^k\| r_2 \le r \|\vec{e}^k\|$$

In general,

$$|e_i^{k+1}| \le \sum_1^{i-1} \left|\frac{A_{ij}}{A_{ii}}\right| |e_j^{k+1}| + \sum_{i+1}^n \left|\frac{A_{ij}}{A_{ii}}\right| |e_j^k|$$

$$\le \sum_1^{i-1} \left|\frac{A_{ij}}{A_{ii}}\right| r \|\vec{e}^k\| + \sum_{i+1}^n \left|\frac{A_{ij}}{A_{ii}}\right| |e_j^k|$$

$$\le \sum_1^{i-1} \left|\frac{A_{ij}}{A_{ii}}\right| \|\vec{e}^k\| + \|e^k\| \sum_{i+1}^n \left|\frac{A_{ij}}{A_{ii}}\right|$$

$$= r_i \|\vec{e}^k\| \le r \|\vec{e}^k\|$$

$\Rightarrow$ true for each i $\Rightarrow$ true for all i $\Rightarrow$

$\|\vec{e}^{k+1}\| \le r \|\vec{e}^k\|$ $r < 1$ $\Rightarrow$ as $k \to \infty$ $\vec{e}^k \to \vec{0}$ $\Rightarrow$ $\vec{x}^k \to \vec{x}^\infty$!