Before proving the spectral theorem, we prove the following lemma:

<u>lemma</u>: Let E be a Hilbert space and $A \in \mathcal{L}(E,E)$.

1) if $A = A^*$ and W is A-invariant ($AW \subseteq W$) then $W^\perp$ is also A-invariant

2) if $A = A^*$ then for all $x \in E$ we have $\langle Ax, x\rangle \in \mathbb{R}$ (in particular, all eigenvalues are real.)

3) $\|A\|_{\mathcal{L}(E,E)} = \sup\{ |\langle Ax, y\rangle| \mid \|x\|_E, \|y\|_E \leq 1\}$

4) if $A = A^*$ then $\|A\|_{\mathcal{L}(E,E)} = \sup\{ |\langle Ax, x\rangle| \mid \|x\|_E \leq 1\}$

5) if $A = A^*$ and $\lambda \neq \mu$ then $E_\lambda \perp E_\mu$

(recall $E_\lambda = \{x \in E \mid Ax = \lambda x\}$ and $\dim E_\lambda > 0$ if $\lambda \in$ point spectrum of A.)

<u>proof</u>:

1) let $y \in W^\perp$ then for any $x \in W$,

$$\langle x, Ay\rangle = \langle Ax, y\rangle \quad \text{since } A^* = A$$

$$= 0 \quad \text{since } AW \subseteq W$$

$\Rightarrow Ay \perp x \ \forall x \in W \Rightarrow Ay \in W^\perp \Rightarrow AW^\perp \subseteq W^\perp$.

2) $\overline{\langle Ax, x\rangle} = \langle x, Ax\rangle$ (rule for inner prods over complex vector spaces)

$= \langle A^* x, x\rangle$

$= \langle Ax, x\rangle$ since $A = A^*$

$\Rightarrow \langle Ax, x\rangle \in \mathbb{R}$.

3) $|\langle Ax, y\rangle| \leq \|Ax\| \, \|y\| \leq \|A\| \, \|x\| \, \|y\|$

$\leq \|A\|$ since $\|x\|, \|y\| \leq 1$

$\Rightarrow \sup\{ |\langle Ax, y\rangle| \;\big|\; \|x\|, \|y\| \leq 1\} \leq \|A\|_{\mathcal{L}(E,E)}$.

Assume $A \neq 0$. (If $A = 0$ then $\|A\| = \sup\{|\langle Ax, y\rangle| \; \|x\|, \|y\| \leq 1\}$ automatically true.)

Let x satisfy $\|x\| \leq 1$.

<u>case 1</u>: $Ax \neq 0$

then $y = \dfrac{Ax}{\|Ax\|}$ satisfies $\|y\| \leq 1$

$\Rightarrow \sup\{ |\langle Ax, y\rangle| \;\big|\; \|x\|, \|y\| \leq 1 \} \geq |\langle Ax, \frac{Ax}{\|Ax\|}\rangle| = \|Ax$

<u>case 2</u>: $Ax = 0$

$\Rightarrow \sup\{ \quad \} \geq \|Ax\|$

$\Rightarrow \sup\{ \; \} \geq \sup\limits_{\|x\| \leq 1} \|Ax\| = \|A\|_{\mathcal{L}(E,E)}$. Combining, $\sup\{\;\} = \|A\|$

4) let $\alpha = \sup \{ |\langle Ax, x\rangle| \mid \|x\| \le 1 \}$

as before $\alpha \le \|A\|$ ⊛

let $\beta = \sup \{ |\langle Ax, y\rangle| \mid \|x\|, \|y\| \le 1 \}$

= smallest # so that

$|\langle Ax, y\rangle| \le \# \, \|x\| \, \|y\| \quad \forall x, y$

from 3), we know $\beta = \|A\|$

So to show $\alpha = \|A\|$, it suffices to show

that $|\langle Ax, y\rangle| \le \alpha \|x\| \, \|y\| \quad \forall x, y$ since

this would imply $\alpha \ge \beta \; (= \|A\|)$

$\Rightarrow \alpha \ge \|A\|$ and then $\alpha = \|A\|$.

We can assume that $\langle Ax, y\rangle$ is a real number.

(why? if not, replace y with $e^{i\alpha} y$ so that $\langle Ax, y\rangle$ becomes real. this will not affect the right hand side $\alpha \|x\| \, \|y\|$.)

I want to write $\langle Ax, y\rangle$ in terms of things like $\langle Aw, w\rangle$. $\langle A(x+y), x+y\rangle = \langle Ax, y\rangle + \langle Ay, x\rangle$

$+ \langle Ax, x\rangle$

$+ \langle Ay, y\rangle$

$$\Rightarrow \langle A(x+y), x+y\rangle = \langle Ax, x\rangle + \langle Ay, y\rangle + 2\langle Ax, y\rangle$$

(used $\langle Ax, y\rangle \in \mathbb{R}$ as well as $A = A^*$!)

Similarly,

$$\langle A(x-y), x-y\rangle = \langle Ax, x\rangle - 2\langle Ax, y\rangle + \langle Ay, y\rangle$$

$$\Rightarrow \langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle = 4\langle Ax, y\rangle$$

$$\rightarrow \quad 4|\langle Ax, y\rangle| \le |\langle A(x+y), x+y\rangle| + |\langle A(x-y), x-y\rangle|$$

$$\le \alpha\|x+y\|^2 + \alpha\|x-y\|^2$$

$$\Rightarrow |\langle Ax, y\rangle| \le \frac{\alpha}{4}\left(\|x+y\|^2 + \|x-y\|^2\right)$$

$$\le \frac{\alpha}{4}\left(2\left(\|x\|^2 + \|y\|^2\right)\right) \qquad \text{parallelogram inequality}$$

$$\Rightarrow |\langle Ax, y\rangle| \le \frac{\alpha}{2}\left(\|x\|^2 + \|y\|^2\right)$$

Now we apply a classic scaling trick

$$\tilde{x} = \sqrt{a}\,x \quad \tilde{y} = y/\sqrt{a} \quad \rightarrow \langle A\tilde{x}, \tilde{y}\rangle = \langle Ax, y\rangle$$

$$\Rightarrow |\langle Ax, y\rangle| \le \frac{\alpha}{2}\left(a\|x\|^2 + \frac{1}{a}\|y\|^2\right) \quad \text{true } \forall a > 0.$$

assume $x \neq 0$.

minimize RHS wrt a. and get that the RHS is smallest when $a = \|y\| / \|x\|$

$\Rightarrow |\langle Ax, y\rangle| \leq \alpha \|x\| \|y\|$. Then we're done!

5) if $Ax = \lambda x$ and $Ay = \mu y$ and $A = A^*$

then $\langle Ax, y\rangle = \langle \lambda x, y\rangle = \lambda\langle x, y\rangle$

$$\| $$

$\langle x, A^* y\rangle = \langle x\ Ay\rangle = \langle x, \mu y\rangle = \bar{\mu}\langle x, y\rangle$

but $A = A^* \Rightarrow \mu = \bar{\mu}$

$\Rightarrow \lambda\langle x, y\rangle - \mu\langle x, y\rangle = 0$

$\Rightarrow \langle x, y\rangle = 0 \Rightarrow E_\lambda \perp E_\mu$. //

Now we'll prove the spectral theorem!

__proof__ If $A = 0$ then any vector is an eigenvector and any orth-normal family spanning E will be our E_0. So assume $A \neq 0$

We want to find an eigenvector.

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We know $\|A\| = \sup\{|\langle Ax, x\rangle| \mid \|x\| \leq 1\}$

since $A = A^*$.

$\Rightarrow$ we have a sequence $\{x_n\} \subseteq E$ with
$\|x_n\| \leq 1$ such that $\langle Ax_n, x_n\rangle \to \|A\|$.

Now, since $\langle Ax, x\rangle \in \mathbb{R}$ $\forall x \in E$, we know that
$\{\langle Ax_n, x_n\rangle\} \subseteq [-\|A\|, \|A\|]$ and $\{|\langle Ax_n, x_n\rangle| \to \|A\|$.
$\Rightarrow$ some subsequence of $\{\langle Ax_n, x_n\rangle\}$ converges to $\|A\|$ or to $-\|A\|$. i.e. some subsequence converges to λ where $|\lambda| = \|A\|$.

We know A is a compact operator $\Rightarrow \{Ax_n\}$ has a convergent subsequence. passing to this subsequence, $Ax_n \to y$ for some $y \in E$.

since $\lambda \neq 0$, we know $y \neq 0$.

Now $\|Ax_n - \lambda x_n\|^2 = \|Ax_n\|^2 - 2\lambda\langle Ax_n, x_n\rangle + \lambda^2\|x_n\|^2$

$$\leq 2\|A\|^2 - 2\lambda\langle Ax_n, x_n\rangle$$

as $n \to \infty$ the RHS $\to 2\|A\|^2 - 2\lambda^2 = 0$

$\therefore \|Ax_n - \lambda x_n\| \to 0$. separately, we know

$\|Ax_n - y\| \to 0$

$\Rightarrow \lambda x_n \to y$ as $n \to \infty$. $\Rightarrow A\lambda x_n \to Ay$ $\Rightarrow Ay = \lambda y$

on the other hand, $Ax_n \to y \Rightarrow A\lambda x_n \to \lambda y$

We've found our first eigenvector!

Now we finish the proof w/ the aid of Zorn's lemma. We look at all sets of orthonormal eigenvectors, with a partial order by inclusion. By Zorn's lemma, $\exists$ a maximal set of orthonormal eigenvectors.

Let $W =$ [span of that maximal orthonormal set]

We want to show $W = E$. i.e $W^{\perp} = \{\vec{0}\}$.

By construction $AW \subseteq W$. So by the lemma, $AW^{\perp} \subseteq W^{\perp}$. $\Rightarrow$ if we restrict A to $W^{\perp}$ then $A|_{W^{\perp}}$ is a self-adjoint bounded linear operator from $W^{\perp}$ to $W^{\perp}$. If $W^{\perp} \neq \{0\}$ then by the argument of the preceding paragraph, $\exists\, x \in W^{\perp}$ so that x is an eigenvector of $A|_{W^{\perp}}$. $\Rightarrow x \perp W$

$\Rightarrow$ our maximal set of orthonormal eigenvectors wasn't maximal. $\cancel{\cdot\cdot}$ $\Rightarrow W^{\perp} = \{0\}$

$\Rightarrow W = E$. $\Rightarrow$ We've found our orthonormal basis of eigenvectors. Now we have two more things to prove. 1) E_{λ} is finite dim

2) given $\varepsilon > 0$, $\{\lambda \mid |\lambda| > \varepsilon\}$ is finite.

1) assume that E_λ is infinite dimensional for some λ. $\Rightarrow$ infinitely many of the ortho-normal eigenvectors are in E_λ.

$$S = \{ i \mid e_i \in E_\lambda \}.$$

then $\{Ae_i\}_{i\in S}$ is an infinite set such that

$$\| Ae_i - Ae_j \|^2 = \lambda^2 + \lambda^2 = 2\lambda^2$$

$\Rightarrow$ no subsequence of $\{Ae_i\}_{i\in S}$ converges $\Rightarrow$ A not compact.

$\times$

2) Similarly, assume $\exists$ $\varepsilon_0 > 0$ so that

$$S = \{ \lambda \mid |\lambda| > \varepsilon_0 \}$$

is infinite for each $\lambda \in S$, let e_λ be an eigenvector. $\Rightarrow$ $\|Ae_\lambda - Ae_\mu\|^2 = \lambda^2 + \mu^2 > 2\varepsilon_0^2$

if $\lambda, \mu \in S$. $\Rightarrow$ $\{Ae_\lambda\}_{\lambda \in S}$ is an infinite set w/ no accumulation point $\Rightarrow$ A not a compact operator. $\times$

this finishes the proof of the spectral theorem //

Corollary: Let $\{A_\alpha\}_{\alpha \in I}$ be a collection of selfadjoint bounded linear operators on a Banach space E so that each A_α is compact and $A_\alpha A_\beta = A_\beta A_\alpha$ for every $\alpha, \beta \in I$. Then $\exists$ an othonormal basis $\{e_j\}$ of E such that e_j is an eigenvector for each A_α.

Note: I'll prove it for two operators A & B $\ni$ A, B are compact and $AB = BA$. Then you see how to generalize

proof First, since A is a compact and self adjoint, by the spectral theorem $\exists$ an othonormal family of eigenvectors of A.

Let E_λ be an eigenspace of A.

$$= \{x \mid Ax = \lambda x\}$$

then $BAx = B\lambda x = \lambda Bx$

but $BAx = ABx \implies A(Bx) = \lambda(Bx)$

$\implies Bx \in E_\lambda$.

i.e $B(E_\lambda) \subseteq E_\lambda$. $B|_{E_\lambda}$ is self adjoint and compact

$\implies$ we can decompose E_λ into orthonormal eigenvectors of B.

$\Rightarrow E_\lambda = \operatorname{span}\{f_j\}_1^N$

where each f_j is an eigenvector of B. Of course since every member of E_λ is an eigenvector of A, we have $\{f_j\}_1^N$ an orthonormal collection of vectors that are eigenvectors of A & of B.

Now take the union over all E_λ

$\Rightarrow$ we have $\displaystyle\bigcup_{\substack{\lambda \in \text{point} \\ \text{spectrum} \\ \text{of } A}} \{f_j\}_1^{N_\lambda}$ is the desired family. //

__Defn:__ Let E be a Hilbert space and $A \in \mathcal{L}(E,E)$. Then A is __normal__ if $AA^* = A^*A$

__ex:__ self adjoint $\Rightarrow$ normal

__ex:__ Unitary ($U^* = U^{-1}$) $\Rightarrow$ normal

(note unitary + compact $\Rightarrow \dim(E) < \infty$).

__Corr:__ Let $A \in \mathcal{L}(E,E)$, E a Hilbert space. A compact & normal. Then E has an orthonormal basis of eigenvectors of A. For each $\lambda \neq 0$, $\dim E_\lambda < \infty$ and $\forall \varepsilon > 0$ $\{\lambda \mid |\lambda| > \varepsilon\}$ is finite.