The first operators we'll consider will be quite friendly. They're the <u>compact</u> operators. (K+F calls them "completely continuous")

<u>defn</u>: Let E and E_1 be Banach spaces. $A: E \to E_1$ a bounded linear functional. Then A is <u>compact</u> if $[AB_1]$ is compact in E_1. (B_1 is the unit ball around $\vec{0}$). Equivalently, $[AB]$ is compact for every bounded set B.

<u>ex</u>: if A has finite rank (range A finite dim.) Then A is a compact operator.

<u>lemma</u>: Assume $\{A_n\} \subseteq \mathcal{L}(E, E_1)$ are compact and $\|A_n - A\|_{\mathcal{L}(E,E_1)} \to 0$ as $n \to \infty$. Then A is compact.

<u>Note</u> this theorem will be very valuable because a standard way to prove that a particular operator is compact is to show it's the limit of a sequence of (well-chosen) compact operators.

proof: to show $[AB_1]$ is compact, we show that AB_1 is totally bounded.

(Recall that a set is compact in a metric space if it's totally bounded & complete. So AB_1 totally bounded $\Rightarrow$ $[AB_1]$ compact.)

i.e. we need to construct an ε-net for AB_1

fix $\varepsilon > 0$ and $x, y \in B_1$. Then

$$\|Ax - Ay\|_{E_1} \le \|Ax - A_n x\|_{E_1} + \|A_n x - A_n y\|_{E_1} + \|A_n y - Ay\|_{E_1}$$

$$\le \|A - A_n\|_{\mathcal{L}(E,E_1)} + \|A_n x - A_n y\|_{E_1}$$

$$+ \|A - A_n\|_{\mathcal{L}(E,E_1)} \quad \text{since } \|x\|_{E}, \|y\|_{E} \le 1$$

take n suff large that $\|A - A_n\|_{\mathcal{L}(E,E_1)} < \varepsilon/3$.

Then for this n, choose a finite $\varepsilon/3$ net for $A_n B_1$

$\{A_n x_1, A_n x_2, \dots A_n x_r\}$.

then $\{Ax_1, \dots Ax_r\}$ is a finite ε-net for A. //

corollary: Let E be a Hilbert space w/ orthonormal basis $\{e_n\}_1^\infty$. Define $A: E \to E$ by $Ae_n = \lambda_n e_n$. Then A is compact if and only if $\lambda_n \to 0$ as $n \to \infty$.

Proof:

($\Leftarrow$) assume $\lambda_n \to 0$. Then for each n we define $A_n : E \to E$ by $A_n\left(\sum_1^\infty \alpha_i e_i\right) = \sum_1^n \alpha_i \lambda_i e_i$

A_n has finite rank $\Rightarrow$ A_n is compact.

Further, $\|A_n - A\|_{\mathcal{L}(E,E)} = \sup\{|\lambda_i| \mid i > n\}$

so $A_n \to A$ $\Rightarrow$ A is compact.

($\Rightarrow$) Assume $\lambda_i \not\to 0$. Then $\exists\, \varepsilon > 0$ so that $S = \{i \mid |\lambda_i| > \varepsilon\}$ is infinite set.

$\Rightarrow$ if $i, j \in S$ then $\|Ae_i - Ae_j\|_E^2 = |\lambda_i|^2 + |\lambda_j|^2 \geq 2\varepsilon^2$

$\Rightarrow \{Ae_i\}_{i \in S}$ is an infinite set w/ no convergent subsequence $\Rightarrow [AB_1]$ is not compact.

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theorem: Let X be a compact space with a finite measure μ. If K is continuous on $X \times X$ then the integral operator

$$T_K f(x) := \int_X K(x,y) f(y)\, d\mu(y)$$

defines a compact operator $T_K : L^2(X) \to L^2(X)$.

Okay... we haven't done measure theory yet. And I haven't defined $L^2(X)$ yet. So for the moment read "dy" instead of "$d\mu(y)$" and for $L^2(X)$ read "the completion of the space of continuous functions on $[0,1]$, where the completion is taken with respect to the L^2-metric."

lemma: the set of linear combinations of separable functions is dense in $C(X \times X)$. i.e. given $K \in C(X \times X)$ and $\varepsilon > 0$, $\exists \{\phi_n\}, \{\psi_n\} \in C(X)$ so that

$$\left| K(x,y) - \sum_{1}^{n} \phi_i(x) \psi_i(y) \right| < \varepsilon$$

proof: Mimic proof of Stone-Weierstrass.

<u>proof of theorem:</u> First, assume K is of the form $K(x,y) = \sum_1^n \phi_i(x)\psi_i(y)$ $\quad \phi_i, \psi_i \in C(X)$.

then $T_K f(x) = \int_X \left(\sum_1^n \phi_i(x)\psi_i(y)\right) f(y)\, d\mu(y)$

$$= \sum_1^n \phi_i(x) \int_X \psi_i(y) f(y)\, d\mu(y)$$

$\Rightarrow$ range (T_K) is finite dimensional.

$\Rightarrow$ T_K is a compact operator.

Now, by lemma, given $K \in C(X,X)$ $\exists$ $\{K_n\}$ of the above form so that $\|K - K_n\|_{L^\infty(X,X)} \to 0$.

We want to show this implies

$\|T_K - T_{K_n}\|_{\mathcal{L}(E,E)} \to 0$ since then

T_K is the limit of compact operators and is then compact.

$$\|T_K f - T_{K_n} f\|_{L^2}^2 = \int_X |T_K f(x) - T_{K_n} f(x)|^2 d\mu(x)$$

$$= \int_X \left| \int_X (K(x,y) - K_n(x,y)) f(y)\, d\mu(y)\right|^2 d\mu(x)$$

$$\Rightarrow \|T_K f - T_{K_n} f\|_{L^2}^2 \le \|K-K_n\|_{L^\infty}^2 \int_X \int_X |f(y)|^2 d\mu(y) d\mu(x)$$

$$= \|K-K_n\|_{L^\infty}^2 \; \mathrm{mass}(X) \int_X |f(y)|^2 d\mu(y)$$

$$\Rightarrow \|T_K f - T_{K_n} f\|_{L^2} \le \sqrt{\mathrm{mass}(X)}\, \|K-K_n\|_{L^\infty} \|f\|_{L^2}$$

$$\Rightarrow \|T_K - T_{K_n}\|_{\mathcal{L}(L^2, L^2)} \le \sqrt{\mathrm{mass}(X)}\, \|K-K_n\|_{L^\infty}$$

and since $\|K-K_n\|_{L^\infty} \to 0$ we have $T_{K_n} \to T_K \Rightarrow T_K$ compact. //

The above proof works for any $L^p(X)$ $1 \le p < \infty$.

<u>Defn</u>: Let E be a Hilbert space w/ orthonormal basis $\{e_n\}_1^\infty$. If $A \in \mathcal{L}(E,E)$ then A is a <u>Hilbert-Schmidt</u> operator if

$$\sum_{i=1}^\infty \sum_{j=1}^\infty |\langle Ae_j, e_i\rangle|^2 < \infty$$

Yikes! That definition depends on the basis $\{e_n\}_1^\infty$. We don't like coordinate-dependent properties. Can A be H-S wrt one basis and not Hilbert-Schmidt with respect to another?

No.

<u>lemma</u>: Let $\{e_n\}_1^\infty$ and $\{f_n\}_1^\infty$ be two orthonormal basis for E. Then

$$\sum_{i,j} |\langle Ae_j, e_i\rangle|^2 = \sum_{i,j} |\langle Af_j, f_i\rangle|^2$$

<u>proof</u>:

$$\sum_j \sum_i |\langle Ae_j, e_i\rangle|^2 = \sum_j \left(\sum_i |\langle Ae_j, e_i\rangle|^2\right)$$

$$= \sum_j \|Ae_j\|^2$$

$$= \sum_j \left(\sum_i |\langle Ae_j, f_i\rangle|^2\right)$$

$$= \sum_j \left(\sum_i |\langle e_j, A^* f_i\rangle|^2\right)$$

$$= \sum_i \left(\sum_j |\langle e_j, A^* f_i\rangle|^2\right) = \sum_i \|A^* f_i\|^2$$

$$\Rightarrow \sum_{i,j} |\langle Ae_j, e_i\rangle|^2 = \sum_i \|A^* f_i\|^2$$

the LHS is indep of $\{f_n\}$ $\Rightarrow$ RHS indep of $\{f_n\}$

the RHS is indep of $\{e_n\}$ $\Rightarrow$ LHS indep of $\{e_n\}$

$$\Rightarrow \sum_{i,j} |\langle Ae_j, e_i\rangle|^2 = \text{some } \# \quad (\text{or } = \infty)$$

$$= \sum_{i,j} |\langle Af_j, f_i\rangle|^2 \checkmark$$

//

<u>defn</u>: if A is Hilbert-Schmidth then we define the Hilbert-Schmidt norm of A:

$$\|A\|_2 = \sqrt{\sum_{i,j} |\langle Ae_j, e_i\rangle|^2}$$

We've just shown that A Hilbert-Schmidt $\Rightarrow$ A^* is Hilbert-Schmidt. and $\|A\|_2 = \|A^*\|_2$.

<u>Q</u>: How does the H-S norm relate to the operator norm?

<u>lemma</u>: if A is Hilbert-Schmidt then

$$\|A\|_{\mathcal{L}(E,E)} \le \|A\|_2$$

Proof: Let $\{e_n\}$ be an orthonormal basis of E.

Let $x \in E$. $\Rightarrow x = \sum_1^\infty c_n e_n$

$$\Rightarrow \|Ax\|_E^2 = \sum_{i=1}^\infty |\langle Ax, e_i\rangle|^2$$

$$= \sum_i |\langle A \sum_j c_j e_j, e_i\rangle|^2$$

$$= \sum_i |\sum_j c_j \langle Ae_j, e_i\rangle|^2$$

$$\leq \sum_i \left(\sum_j |c_j| |\langle Ae_j, e_i\rangle|\right)^2$$

$$\leq \sum_i \left(\sum_j |c_j|^2\right)\left(\sum_j |\langle Ae_j, e_i\rangle|^2\right)$$

$$= \sum_j |c_j|^2 \sum_i \sum_j |\langle Ae_j, e_i\rangle|^2$$

$$= \|x\|_E^2 \, \|A\|_2^2$$

$$\Rightarrow \|Ax\|_E \leq \|A\|_2 \, \|x\|_E$$

$$\Rightarrow \|A\|_{\mathcal{L}(E,E)} \leq \|A\|_2 \quad //$$

<u>Theorem</u>: $A : E \to E$, E Banach spaces.
If A is Hilbert-Schmidth then A is compact.

<u>proof</u>: Let $\{e_n\}$ be an orthonormal basis
For each n, define $A_n \in \mathcal{L}(E,E)$ by

$$A_n(x) = \sum_1^n c_j A_r e_j .$$

As before, range A_n = finite dimensional.
$\Rightarrow A_n$ is compact operator. Also,
$A - A_n$ is Hilbert-Schmidt and

$$\|A - A_n\|_2^2 = \sum_{n+1}^\infty \|Ae_i\|^2 < \infty$$

Since $\sum_1^\infty \|Ae_i\|^2 < \infty$ we know the tail $\to 0$ as $n \to \infty$

$\Rightarrow \|A - A_n\|_2 \to 0$ as $n \to \infty$ And since

$\|A - A_n\|_{\mathcal{L}(E,E)} \leq \|A - A_n\|_2$ we know

$A_n \to A$ as operators $\Rightarrow A$ is compact. //

Proposition: Let X be a measure space and $K \in L^2(X \times X)$. Then $T_K : L^2(X) \to L^2(X)$ defined by

$$T_K f(x) = \int_X K(x,y) f(y) \, d\mu(y)$$

is Hilbert-Schmidt, hence compact.

Proof: trivial once we know that $L^2(X)$ is a Hilbert space. Which it is...

Proposition: Let E, E_1, and E_2 be Banach spaces and

$$A : E \to E_1 \quad , \quad B : E_1 \to E_2$$

be bounded linear operators. Then $B \circ A$ is compact if A is compact or B is compact.

proof: simple exercise from defn of compact.

Recall that if E is finite dimensional then $A : E \to E$ self adjoint

$\Rightarrow$ A has an orthonormal basis of eigenvectors

Note: in infinite dimensions, $\exists$ self adjoint bounded operators that have no eigenvalues. For example, Take

$E = L^2([0,1])$ = completion of continuous functions on $[0,1]$ with respect to L^2 metric

define $M_x : E \to E$ by

$(M_x f)(x) = x f(x)$ (multiply $f(x)$ by x.)

M_x is self adjoint. And if $\lambda \neq 0$ is an eigenvalue then $M_x f(x) = x f(x) = \lambda f(x)$ almost all $x \in [0,1]$

$\Rightarrow (x - \lambda) f(x) = 0$ $\forall$ almost all $x \in [0,1]$

$\Rightarrow f(x) = 0$ almost all $x \in [0,1]$

$\Rightarrow f = \vec{0}$ $\Rightarrow$ λ not an eigenvalue

<u>Spectral theorem for compact operators</u>:

Let E be a Hilbert space and $A \in \mathcal{L}(E,E)$ be compact and self-adjoint. Then E has an orthonormal basis ~~a~~ consisting of eigenvectors of A. Furthermore, for each nonzero eigenvalue λ, the subspace

$$E_\lambda = \{ x \in E \mid Ax = \lambda x \}$$

is finite-dimensional <u>and</u> for each $\varepsilon > 0$

$\{ \lambda \mid |\lambda| \geq \varepsilon \text{ and } \dim E_\lambda > 0 \}$ is finite.

<u>transl</u>: if you fix a ball around $0 \in \mathbb{C}$ and ask how many eigenvalues are outside this ball, the answer is: finitely many.

<u>transl</u>: if $\{\lambda_n\}_1^\infty$ is the collection of eigenvalues then the only limit point of this set is $0 \in \mathbb{C}$.