Measure Theory.

We will first construct the Lebesgue measure in $\mathbb{R}^n$ and we will then move on to abstract measure theory.

In $\mathbb{R}^n$, we know what the measure of certain sets should be. Specifically, we have a confident answer for the volume of a rectangle, for the volume of a ball, for the volume of a line (0), for the volume of a point (0)

We would like to talk about the volume of other sets in $\mathbb{R}^n$ as well. Our definition of volume should satisfy some properties like

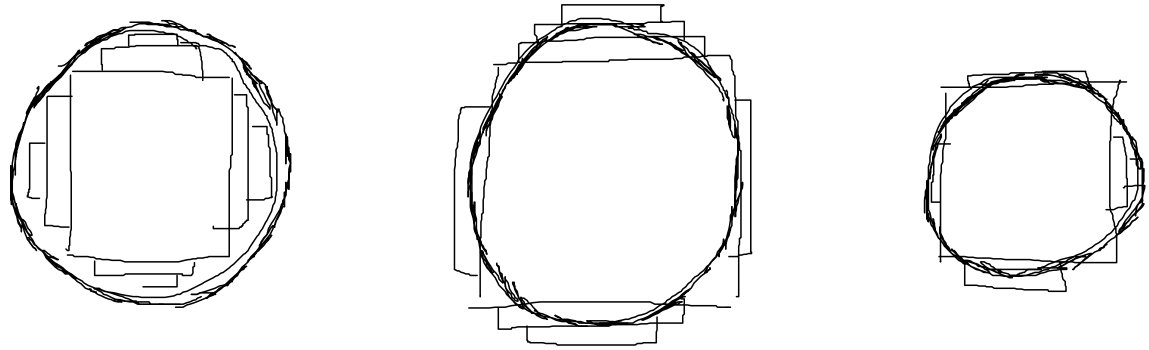
- translation invariance
- rotation invariance
- scales properly w/ dilation
- agrees w/ known facts like volume of rectangle, volume of line, volume of point.

Good news: We can construct such a thing. We call it a "measure" and $\mu(E) \in [0, \infty]$ is the measure (aka volume) of E.

Bad news: $\exists$ sets $E \subseteq \mathbb{R}^n$ that cannot be measured. :(

Our construction will be based on the volume of rectangles. Specifically, we'll make sure that we assign rectangles the correct volume and then we'll go from there. How? We approximate sets w/ rectangles.

Q: Consider E = unit disc. How do you want to approximate E?



from within? from without? randomly?

Okay, randomly seems like a dopey idea.

From within will always underestimate the volume, but it might be a good idea to define

inner measure of E = sup {measure of app by rect.}
taken over all approximations from within

outer measure of E = inf {measure of app. by rectangles}
taken over all approximations from without.

Certainly, if

inner measure of E = outer measure of E

we'd be very confident about our measure.

In fact, we can make do just by using outer measure (why?!?) as we will see.

So much for religion*, on to nuts & bolts.

<u>defn</u>: A <u>rectangle</u> in $\mathbb{R}^n$ is a subset I of $\mathbb{R}^n$ which can be written as a Cartesian product

$$I = \prod_1^n [a_k, b_k]$$

of closed intervals $[a_k, b_k]$ where $a_k \le b_k$ for each $k = 1..n$

<u>transl</u>: $\vec{x} \in I \iff a_k \le x_k \le b_k \quad \forall k = 1...n$

given a rectangle I, we define its diameter and volume:

$$\text{diam}(I) = \sqrt{\sum_1^n (b_k - a_k)^2}, \quad \text{vol}(I) = \prod_1^n (b_k - a_k)$$

<u>Note</u>: we'll let $\emptyset$ be a rectangle too. Its diameter and volume $= 0$.

* note: how would this construction yield rotation invariance?!?

defn: Given a collection $\mathcal{C}$ of rectangles, we say that $\mathcal{C}$ is nonoverlapping if $I_1 \in \mathcal{C}, I_2 \in \mathcal{C} \Rightarrow (I_1)^\circ \cap (I_2)^\circ = \phi$.

i.e. disjoint interiors.

lemma 1.1.1 1) If $\mathcal{C}$ is a non-overlapping finite collection of rectangles, each of which is contained in the rectangle J then

$$\mathrm{vol}(J) \geq \sum_{I \in \mathcal{C}} \mathrm{vol}(I)$$

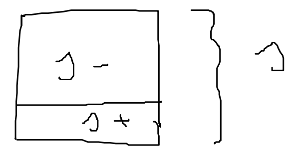
2) If $\mathcal{C}$ is a finite collection of rectangles and J is a rectangle covered by $\mathcal{C}$ $\left(J \subseteq \bigcup_{I \in \mathcal{C}} I\right)$

then $$\mathrm{vol}(J) \leq \sum_{I \in \mathcal{C}} \mathrm{vol}(I).$$

Now that seems obvious! Let's prove it.

proof: WLOG, assume $I \in \mathcal{C} \Rightarrow I \subseteq J$. This is certainly the case in 1). In 2) if $I \in \mathcal{C}$ and $I \not\subseteq J$ then replace I with $I \cap J$. This will still be a rectangle (why?) and will only decrease the area (why?) So if we can prove 2) for $I \cap J$ we're okay...

Now, if $J = \prod_1^n [a_k, b_k]$ and $a_k \le c \le b_k$ for some k and J^+ is the rectangle obtained by replacing $[a_k, b_k]$ with $[a_k, c]$ and J^- is the rectangle obtained by replacing $[a_k, b_k]$ with $[c, b_k]$ then $\text{vol}(J) = \text{vol}(J^+) + \text{vol}(J^-)$



In general if you cut $[a_k, b_k]$ up into n_k pieces

$a_k = c_{k0} \le c_{k1} \le \dots \le c_{kn_k} = b_k$ then the n_k new rectangles will have their volumes sum correctly. Further, you can cut the n-sides of J all at once (not focussing just on the k^{th} side) to obtain a collection $\mathcal{S}$ of $\prod_1^n n_k$ rectangles of the form

$$R(m_1 \dots m_n) = \prod_1^n [c_{k,m_k-1}, c_{k,m_k}]$$

and $\text{vol}(J) = \sum_{R \in \mathcal{S}} \text{vol}(R)$

Now return to our collection $\mathcal{C}$ which is a finite collection of rectangles I contained in J. We use each I to <u>choose</u> the partitioning c_{k,m_k} of the intervals $[a_k, b_k]$

Specifically, given $I \in \mathcal{C}$, since I is a rectangle, $I = \prod_1^n [\alpha_k, \beta_k]$ for some $\alpha_k \le \beta_k$.

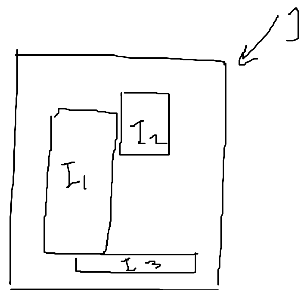
Since $I \subseteq J$ we know $a_k \le \alpha_k$ and $\beta_k \le b_k$.

So let $c_{k1} = \alpha_k$ $c_{k2} = \beta_k$ $c_{k3} = b_k$.

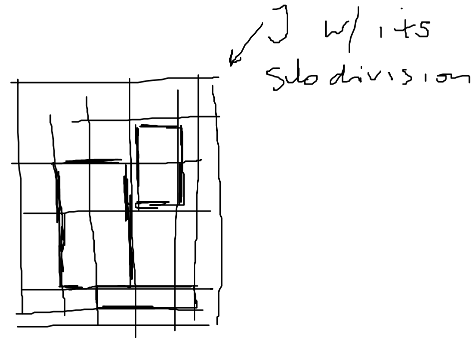
Now take the second rectangle in $\mathcal{C}$. It has its own sides (its own α_k + β_k). Insert α_k and β_k into

$$a_k = c_{k0} \le c_{k1} \le c_{k2} \le c_{k3} = b_k$$

w/o breaking the ordering. And so on. In this way, you take your finite $\mathcal{C}$ and create a finite division of J. Note



will yield



So given $\mathcal{C}$, we construct Δ and we know that

$$\mathrm{vol}(J) = \sum_{R \in \Delta} \mathrm{vol}(R)$$

and we know that for each $I \in \mathcal{C}$ there is a subcollection of Δ, call it $\Delta(I)$ so that

$$\mathrm{vol}(I) = \sum_{R \in \Delta(I)} \mathrm{vol}(R)$$

$\Delta(I) = \{R \in \mathcal{C} \mid R \subseteq I\}$.

Now add the assumption that the members of $\mathcal{C}$ are <u>non overlapping</u>.

Let I and I' be distinct members of $\mathcal{C}$. if $R \in \Delta(I)$ and $R \in \Delta(I')$ then

$$R = \prod_{1}^{n} [c_{k,m_k-1}, c_{k,m_k}]$$

when $c_{k,m_k-1} = c_{k,m_k}$ for some k i.e. R is contained in a face of I and I'. $\Rightarrow \text{vol}(R) = 0$.

Thus
$$\text{vol}(J) = \sum_{R \in \Delta} \text{vol}(R) \geq \sum_{I \in \mathcal{C}} \sum_{R \in \Delta(I)} \text{vol}(R) = \sum_{I \in \mathcal{C}} \text{vol}(I)$$

and done! We've shown that if you have a nonoverlapping finite family of rectangles contained in J then
$$\text{vol}(J) \geq \sum_{I \in \mathcal{C}} \text{vol}(I).$$

Now to show that given a finite family of rectangles so that $J \subseteq \bigcup_{I \in \mathcal{C}} I$ then $\text{vol}(J) \leq \sum_{I \in \mathcal{C}} \text{vol}(I)$.

First, if $I \in \mathcal{C}$ and $I \nsubseteq J$ then $I \cap J$ is a rectangle as is $I \cap J^c$.

So $\operatorname{vol}(I) = \operatorname{vol}(I \cap J) + \operatorname{vol}(I \cap J^c)$

$\rightarrow \operatorname{vol}(I \cap J) \leq \operatorname{vol}(I)$. and $J \subseteq \bigcup_{I \in \mathcal{C}} (I \cap J)$

It suffices, therefore, to prove

$$\operatorname{vol}(J) \leq \sum_{I \in \mathcal{C}} \operatorname{vol}(I \cap J).$$

So we're in the situation we discussed earlier, where we assume (WLOG) that $I \subseteq J$.

If I and I' overlap then $\exists R \in \mathcal{S}(I)$ s.t. that $R \in \mathcal{S}(I')$ and $\operatorname{vol}(R) \neq 0$

$\Rightarrow$ if we sum over $\mathcal{S}(I)$ and $\mathcal{S}(I')$ then we count R twice, increasing the sum.

$$\operatorname{vol}(J) = \sum_{R \in \mathcal{S}} \operatorname{vol}(R) \leq \sum_{I \in \mathcal{C}} \sum_{R \in \mathcal{S}(I)} \operatorname{vol}(R) = \sum_{I \in \mathcal{C}} \operatorname{vol}(I)$$

and done! //

note. This step uses both $J = \bigcup_{I \in \mathcal{C}} I$ and the possible overlapping.

Ouch! That was a fair bit of work to prove the obvious!

We are now in a position to define outer Lebesgue measure.

Let $\Gamma \subseteq \mathbb{R}^n$ be a subset. Let $\mathcal{C}$ be a countable possibly overlapping cover of Γ by rectangles

$$\Gamma \subseteq \bigcup_{I \in \mathcal{C}} I$$

define $\Sigma(\mathcal{C}) := \sum_1^\infty vol(I_i) \in [0, \infty]$.

(given a cover, add up the volumes of the rectangles.)

defn: The <u>outer Lebesgue measure of Γ</u> is

$$|\Gamma|_e = \inf \left\{ \Sigma(\mathcal{C}) \;\middle|\; \Gamma \subseteq \bigcup_{I \in \mathcal{C}} I \quad \mathcal{C} \text{ is countable, each member of } \mathcal{C} \text{ is a rectangle.} \right\}$$

<u>Note</u>: rectangles have finite volume since we required them to be of the form

$$\prod_1^n [a_k, b_k].$$

But we can cover an unbounded object any way. for example, if in $\mathbb{R}$

$$\Gamma = \bigcup_1^\infty \left[n, n + \tfrac{1}{2^n}\right) \quad \text{then} \quad \Gamma \text{ is covered both}$$

by $\mathcal{C}_1 = \{ [n, n+1] \mid n \in \mathbb{N} \}$ and by $\mathcal{C}_2 = \{ [n, n + \frac{1}{2^n}] \mid n \in \mathbb{N} \}$

$$\Sigma(\mathcal{C}_1) = \infty \qquad \Sigma(\mathcal{C}_2) < \infty.$$

We will find that in order to describe integration in a reasonable manner, that our concept of measure will need to satisfy:

$$\circledast \quad \mu\left(\bigcup_1^\infty \Gamma_i\right) = \sum_1^\infty \mu(\Gamma_i) \qquad \text{if } \Gamma_i \cap \Gamma_j = \phi \text{ for } i \neq j$$

i.e. μ is "countably additive". This is a very useful property for a number of reasons. For example, if $\{A_k\}$ is a collection of measurable sets such that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$

then you would like

$$\mu\left(\bigcap_1^\infty A_k\right) = \lim_{k\to\infty} \mu(A_k). \qquad \circledast\circledast$$

To prove this, you need $\circledast$. Note: if $\circledast\circledast$ holds then a single point in $\mathbb{R}^n$ $n \geq 1$ will have measure zero (if the measure gives the right answer for rectangles) and the measure of a face of a rectangle will have measure 0 too.

So even though our definition of outer measure is defined for arbitrary subsets of $\mathbb{R}^n$, we are going to have to look at a restricted collection of subsets

$$\overline{\mathcal{B}_{\mathbb{R}^n}}$$

so that the map

$$\Gamma \in \overline{\mathcal{B}_{\mathbb{R}^n}} \longrightarrow |\Gamma|_e$$

satisfies the countable additivity property.

First, we'll check that outer measure does what it should do if Γ is a rectangle.

Lemma 2.1.1 if $\Gamma = \bigcup_1^m J_k$ where the J_k's are non overlapping rectangles then

$$|\Gamma|_e = \sum_1^m \mathrm{vol}(J_k).$$

Corr: $|J|_e = \mathrm{vol}(J)$ if J is a rectangle.

proof: By definition of $|\Gamma|_e$, we know that $|\Gamma|_e \leq \sum_1^m \mathrm{vol}(J_k)$. We now prove the opposite direction by showing that if $\mathcal{C}$ is a countable cover by rectangles then $\Sigma(\mathcal{C}) \geq \sum_1^m \mathrm{vol}(J_k)$

Fix $\varepsilon > 0$

Let $\mathcal{C}$ be a cover of J by rectangles.

$\mathcal{C} = \{I_\ell\}$. For each I_ℓ

create a slightly larger rectangle I'_ℓ so that

1) $I_\ell \subseteq (I'_\ell)^\circ$

2) $\mathrm{vol}(I'_\ell) \le \mathrm{vol}(I_\ell) + \frac{\varepsilon}{2^\ell}$

We know Γ is the finite union of rectangles, each of which is compact $\Rightarrow \Gamma$ is compact

$\therefore \exists$ a finite subcollection of $\{I'_\ell\}$ that covers Γ.

$\Rightarrow \Gamma$ is covered by $\{I'_1 \dots I'_L\}$.

$\Rightarrow$ by lemma 1.1.1,

$$\sum_1^m \mathrm{vol}(J_k) \le \sum_{k=1}^m \sum_{\ell=1}^L \mathrm{vol}(J_k \cap I'_\ell)$$

$$= \sum_{\ell=1}^L \underbrace{\sum_{k=1}^m \mathrm{vol}(J_k \cap I'_\ell)}_{\text{since } J_k \cap I'_\ell \text{ is a collection of disjoint rectangles in } I'_\ell}$$

$$\le \sum_{\ell=1}^L \mathrm{vol}(I'_\ell) \le \sum_{\ell=1}^L \mathrm{vol}(I_\ell) + \frac{\varepsilon}{2^\ell}$$

$$\le \sum_1^\infty \mathrm{vol}(I_\ell) + \frac{\varepsilon}{2^\ell} = \Sigma(\mathcal{C}) + \varepsilon.$$

Now, take $\varepsilon \downarrow 0$ and $\sum_1^m \mathrm{vol}(J_k) \le \Sigma(\mathcal{C})$ as desired. //

This proves that if I is a rectangle then

$$\text{vol}(I) = |I|_e.$$

So far so good... our exterior measure respects rectangles.

For countable additivity, we can get one side of the inequality for all subsets of $\mathbb{R}^n$.
i.e. if Γ and $\{\Gamma_m\}$ are arbitrary subsets of $\mathbb{R}^n$ with

$$\Gamma \subseteq \bigcup_1^\infty \Gamma_m \quad \text{then} \quad |\Gamma|_e \leq \sum_1^\infty |\Gamma_m|_e$$

it's the other direction

$$|\Gamma|_e \geq \sum_1^\infty |\Gamma_m| \quad \text{if } \{\Gamma_m\} \text{ pairwise disjoint and } \Gamma = \bigcup_1^\infty \Gamma_m$$

that will require additional constraints on the subsets Γ, Γ_m of $\mathbb{R}^n$.

<u>lemma 2.1.2</u>: if $\Gamma_1 \subseteq \Gamma_2$ then $|\Gamma_1|_e \leq |\Gamma_2|_e$. if $\Gamma \subseteq \bigcup_1^\infty \Gamma_m$ then $|\Gamma|_e \leq \sum_1^\infty |\Gamma_m|_e$. In particular, if $\Gamma \subseteq \bigcup_1^\infty \Gamma_m$ where $|\Gamma_m|_e = 0$ for each m then $|\Gamma|_e = 0$. and therefore $|\partial I|_e = 0$ for any rectangle I. $(\partial I := [I] - I^\circ.)$
Finally, if $\Gamma_1, \Gamma_2 \subseteq \mathbb{R}^n$ have $\text{dist}(\Gamma_1, \Gamma_2) > 0$ then

$$|\Gamma_1 \cup \Gamma_2|_e = |\Gamma_1|_e + |\Gamma_2|_e$$

proof:

1) $\Gamma_1 \subseteq \Gamma_2 \Rightarrow |\Gamma_1|_e \le |\Gamma_2|_e$ is automatic since any countable covering of rectangles that covers Γ_2 certainly covers Γ_1. $\Rightarrow |\Gamma_1|_e$ is infimum over a larger set $\Rightarrow |\Gamma_1|_e \le |\Gamma_2|_e$.

2) Fix $\varepsilon > 0$. For each Γ_n, choose a covering $\mathcal{C}_n$ so that $\Sigma(\mathcal{C}_n) \le |\Gamma_n|_e + \varepsilon/2^n$. Then $\bigcup_1^\infty \mathcal{C}_n$ is a countable covering of rectangles that covers $\bigcup_1^\infty \Gamma_n$.

Hence it covers Γ. $\Rightarrow$

$$|\Gamma|_e \le \Sigma(\mathcal{C}) = \Sigma\left(\bigcup_1^\infty \mathcal{C}_m\right)$$

$$= \sum_{n=1}^\infty \sum_{I \in \mathcal{C}_n} \operatorname{vol}(I)$$

$$\le \sum_{n=1}^\infty |\Gamma_n|_e + \varepsilon/2 = \sum_1^\infty |\Gamma_n|_e + \varepsilon.$$

Since ε was arbitrary, $|\Gamma|_e \le \sum_1^\infty |\Gamma_m|_e$

3) If I is a rectangle then $|\partial I|_e = 0$. because ∂I is the finite union of degenerate rectangles

$$[a_1,b_1] \times [a_2,b_2] \times \dots \times \{c_k\} \times \dots [a_n,b_n]$$

each of which has outer measure 0.

4) Let $\delta = \text{dist}(\Gamma_1, \Gamma_2)$.

Let $\mathcal{C}$ be a countable cover by rectangles of $\Gamma_1 \cup \Gamma_2$. WLOG, assume that each rectangle in $\mathcal{C}$ has diameter $< \delta$. (Take any rectangle w/ diam $\geq \delta$ and divide it into smaller rectangles (nonoverlapping) so that vol(big rectangle) $= \sum_i^L$ vol(smaller ones).)

Now, given $\mathcal{C}$ define

$$\mathcal{C}_1 = \{ I \in \mathcal{C} \mid I \cap \Gamma_1 \neq \phi \}$$

$$\mathcal{C}_2 = \{ I \in \mathcal{C} \mid I \cap \Gamma_2 \neq \phi \}.$$

Since $\text{dist}(\Gamma_1, \Gamma_2) = \delta$, and $I \in \mathcal{C} \Rightarrow \text{diam}(I) < \delta$ we know that $I \in \mathcal{C}_1 \Rightarrow I \notin \mathcal{C}_2$ and

$I \in \mathcal{C}_2 \Rightarrow I \notin \mathcal{C}_1$

$\mathcal{C}_1$ is a covering by rectangles of Γ_1

$$\Rightarrow |\Gamma_1|_e \leq \Sigma(\mathcal{C}_1)$$

Similarly, $|\Gamma_2|_e \leq \Sigma(\mathcal{C}_2)$.

and $\Sigma(\mathcal{C}_1) + \Sigma(\mathcal{C}_2) \leq \Sigma(\mathcal{C})$ since there might be rectangles in $\mathcal{C}$ that aren't in $\mathcal{C}_1$ or $\mathcal{C}_2$.

therefore we've shown

$$|\Gamma_1|_e + |\Gamma_2|_e \leq \Sigma(\mathcal{C})$$

where $\mathcal{C}$ was a countable covering by rectangles of $\Gamma_1 \cup \Gamma_2$. Taking the infimum over all such $\mathcal{C}$, we get

$$|\Gamma_1|_e + |\Gamma_2|_e \leq |\Gamma_1 \cup \Gamma_2|_e$$

Now to show the other direction. $|\Gamma_1 \cup \Gamma_2|_e \leq |\Gamma_1|_e + |\Gamma_2|_e$

Let $\mathcal{C}_1$ cover Γ_1 and $\mathcal{C}_2$ cover Γ_2.

$\Rightarrow$ $\mathcal{C}_1 \cup \mathcal{C}_2$ covers $\Gamma_1 \cup \Gamma_2$

$$\Rightarrow \quad |\Gamma_1 \cup \Gamma_2|_e \leq \Sigma(\mathcal{C}_1 \cup \mathcal{C}_2) = \Sigma(\mathcal{C}_1) + \Sigma(\mathcal{C}_2)$$

take the infimum over all $\mathcal{C}_1$ and $\mathcal{C}_2$

$$\Rightarrow \quad |\Gamma_1 \cup \Gamma_2|_e \leq |\Gamma_1|_e + |\Gamma_2|_e.$$

$\Rightarrow$ $\text{dist}(\Gamma_1, \Gamma_2) > 0 \Rightarrow |\Gamma_1 \cup \Gamma_2|_e = |\Gamma_1|_e + |\Gamma_2|_e$.

as desired. //

Okay, now we want to relate our coverings by rectangles to the topology on $\mathbb{R}^n$. This should be do-able since the (standard metric) topology is equivalent to the ℓ^∞-metric topology which has rectangles as open sets.

Let

$\mathcal{G}$ = open sets in $\mathbb{R}^n$ (usual metric topology)

$\mathcal{G}_\delta$ = all sets that can be written as a countable intersection of members of $\mathcal{G}$

$$\bigcap_1^\infty G_m, \quad G_m \in \mathcal{G}$$

$\mathcal{F}$ = closed sets in $\mathbb{R}^n$

$\mathcal{F}_\sigma$ = all countable unions of closed sets

$$\bigcup_1^\infty F_m \quad F_m \in \mathcal{F}.$$

fact: $\mathcal{G} \cup \mathcal{F} \subseteq \mathcal{G}_\delta$

$\Gamma \in \mathcal{G}_\delta \Rightarrow \Gamma^c \in \mathcal{F}_\sigma$

Lemma 2.1.4 Let $\Gamma \subseteq \mathbb{R}^n$.

Then $$|\Gamma|_e = \inf \{ |G|_e \mid \Gamma \subseteq G \in \mathcal{G} \}$$

In particular, given $\Gamma \subseteq \mathbb{R}^n$ $\exists$

$B \in \mathcal{G}_\delta$ so that $|\Gamma|_e = |B|_e$ and $\Gamma \subseteq B$.

this tells us that to find the outer measure of an arbitrary set Γ, we look at all open sets containing Γ and take the inf of their outer measures

Note: the theorem does not say that $\Gamma \in \mathcal{G}_\delta$. It says that $\exists$ a $\mathcal{G}_\delta$ set containing Γ whose outer measure equals Γ's.

proof:

First, since $\Gamma \subseteq G \Rightarrow |\Gamma|_e \leq |G|_e$, it's clear that

$$|\Gamma|_e \leq \inf\{ |G|_e \mid \Gamma \subseteq G \in \mathcal{G}\},$$

also, if $|\Gamma|_e = \infty$ then $|\Gamma|_e = \inf\{\ \}$ will be true.

So assume $|\Gamma|_e < \infty$. Fix $\varepsilon > 0$. Choose a countable covering by rectangles of Γ so that

$$\Sigma(\mathcal{C}) \leq |\Gamma|_e + \varepsilon/2.$$

$\mathcal{C} = \{I_k\}_1^\infty$. Now, for each I_k, swell it a little, so that I_k' is a rectangle satisfying

$$I_k \subseteq (I_k')^\circ$$

and $|I_k'|_e \leq |I_k|_e + \frac{\varepsilon}{2^{k+1}}$ used $(vol(I_k) = |I_k|_e$ here!)

then $G = \bigcup_1^\infty (I_k')^\circ$ is an open set that contains Γ. Further,

$$|G|_e \leq \Sigma(\mathcal{C}') = \sum_1^\infty vol(I_k') = \sum_1^\infty |I_k'|_e$$
$$\leq \sum_1^\infty |I_k|_e + \frac{\varepsilon}{2} = \Sigma(\mathcal{C}) + \frac{\varepsilon}{2}$$

So we fixed ε, found $\mathcal{C}$ so that

$$\Sigma(\mathcal{C}) \le |\Gamma|_e + \varepsilon/2$$

and then found $G \in \mathcal{G}$ so that

$$|G|_e \le \Sigma(\mathcal{C}) + \varepsilon/2$$

combining, $|G|_e \le |\Gamma|_e + \varepsilon$.

Since ε was arbitrary, this shows

$$\inf\{ |G|_e \mid \Gamma \subseteq G \in \mathcal{G} \} \le |\Gamma|_e$$

$\therefore$ $|\Gamma|_e = \inf\{ \}$, as desired.

Now, to construct the $\mathcal{G}_\delta$ set... Given $\Gamma \subseteq \mathbb{R}^n$, choose $G_n \in \mathcal{G}$ so that $\Gamma \subseteq G_n$

$$|G_n| \le |\Gamma|_e + \frac{1}{n}$$

let $B = \bigcap_1^\infty G_n$ then $B \in \mathcal{G}_\delta$, $\Gamma \subseteq B$, and

$$|B|_e \le |G_n|_e \text{ for each } n.$$

$\Rightarrow$ $|B|_e = |\Gamma|_e$ as desired. //

Okay, now we're ready to define our mystery set $\overline{\mathcal{B}}_{\mathbb{R}^n}$ (the collection of subsets of $\mathbb{R}^n$ on which our outer measure is countably additive.)

defn: $\Gamma \subset \mathbb{R}^n$ is Lebesgue measurable

i.e. $\Gamma \in \overline{\mathcal{B}_{\mathbb{R}^n}}$ if for each $\varepsilon > 0 \ \exists \ G_\varepsilon \in \mathcal{G}$

so that $\Gamma \subseteq G_\varepsilon$ and $|G_\varepsilon - \Gamma|_e < \varepsilon$.

defn: Lebesgue measure

$$|\Gamma| \in [0, \infty]$$

is defined for each $\Gamma \in \overline{\mathcal{B}_{\mathbb{R}^n}}$.

by $|\Gamma| = |\Gamma|_e$.

i.e outer measure = Lebesgue measure if $\Gamma \in \overline{\mathcal{B}_{\mathbb{R}^n}}$

What if $\Gamma \notin \overline{\mathcal{B}_{\mathbb{R}^n}}$? Well, in that case the outer measure $|\Gamma|_e$ is defined but Lebesgue measure isn't.

Q: How do we know Lebesgue measure is countably-additive?

Q: are there any sets that aren't Lebesgue measurable?

here's some bad logic for you: Let G be an open set containing Γ. Since

$$|G|_e = |\Gamma|_e + |G - \Gamma|_e$$

and since by lemma 2.1.4 $|\Gamma|_e = \inf\{|G|_e\}$, it

follows that we can make
$|G-\Gamma|_e$ as small as we
want because $|G-\Gamma|_e = \underbrace{|G|_e - |\Gamma|_e}_{\text{gets as small as we want.}}$

$\therefore$ given $\varepsilon > 0$ $\exists$ G_ε w/ $|G-\Gamma|_e < \varepsilon \Rightarrow \Gamma$ is Lebesgue measurable!

What was our mistake here ?!?

We don't know $|G|_e = |G-\Gamma|_e + |\Gamma|_e$!!! (ooh!)
We certainly proved this if G and Γ are rectangles
We also proved it if $\text{dist}(G,\Gamma) > 0$ But in general, the best we can prove is

$$|G|_e \le |G-\Gamma|_e + |\Gamma|_e$$

and our bad-logic proof that all sets are Lebesgue measurable breaks down.

In fact, $\exists$ sets that are not Lebesgue measurable.

Good news: every open set is Lebesgue measurable
Good news: if $|\Gamma|_e^{\kappa} = 0$ then Γ is Lebesgue measurable.
Why? given $\varepsilon > 0$, choose $G \in \mathcal{G}$ so that $|G|_e < \varepsilon$ $\Gamma \subseteq G$
$\Rightarrow |G-\Gamma|_e \le |G|_e < \varepsilon$.

Good news:

if Γ is Lebesgue measurable then $\exists\, B \in \mathcal{G}_\delta$ so that $\Gamma \subseteq B$ and $|B-\Gamma|_e = 0$

Why? for each n, $\exists\, G_n \in \mathcal{G}$ so that

$$|G_n - \Gamma|_e < 1/n$$

$\Rightarrow$ if $B = \bigcap_1^\infty G_n$ then $|B-\Gamma|_e \leq |G_n - \Gamma|_e < \frac{1}{n}$ $\forall n$

$\Rightarrow$ $|B-\Gamma|_e = 0$ and $\Gamma \subseteq B \in \mathcal{G}_\delta$.

We want to show that $\overline{\mathcal{B}_{\mathbb{R}^n}}$ is a large collection of sets. First, we'll show it's closed under countable unions. Then we'll show it's closed under complementation.

Lemma 2.1.7: Let $\{\Gamma_k\}_1^\infty \subseteq \overline{\mathcal{B}_{\mathbb{R}^n}}$. Then

$$\Gamma = \bigcup_1^\infty \Gamma_k \in \overline{\mathcal{B}_{\mathbb{R}^n}} \quad \text{and} \quad |\Gamma| \leq \sum_1^\infty |\Gamma_n|$$

proof: For each n, choose $G_n \in \mathcal{G}$ so that $\Gamma_n \subseteq G_n$ and $|G_n - \Gamma_n|_e < \varepsilon/2^n$. Then $G = \bigcup_1^\infty G_n \in \mathcal{G}$, it contains Γ and by lemma 2.1.2 satisfies

$$|G-\Gamma|_e \leq \left|\bigcup_1^\infty (G_n - \Gamma_n)\right|_e \leq \sum_1^\infty |G_n - \Gamma_n|_e < \varepsilon.$$

<u>Fact</u>: a rectangle is Lebesgue measurable since

$$I = I^\circ \cup \partial I$$

It's the union of Lebesgue measurable sets

(I° is L. measurable since open.
∂I is L. measurable since $|\partial I|_e = 0$)

Okay, we've shown that $\overline{\mathcal{B}_{\mathbb{R}^n}}$ is closed under countable unions. Now we want to show that it's closed under complements.

To do this, we want the multi-dim version of the fact we know from $\mathbb{R}$: "If U is an open set (usual metric topology) in $\mathbb{R}$ then U is the union of a countable set of mutually disjoint intervals".

<u>lemma 2.1.9</u>: If $G \in \mathcal{G}$, $G \subseteq \mathbb{R}$ then G is the union of a countable number of mutually disjoint open intervals. More generally, if G is an open set in $\mathbb{R}^n$ then for each $\delta > 0$ G admits a countable non-overlapping exact cover by cubes Q with $\operatorname{diam}(Q) < \delta$

proof:

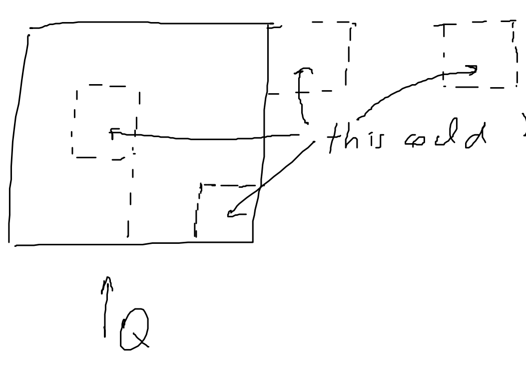
1) Let $G \subset \mathbb{R}$ be open. For each $x \in G$, let $\mathring{I}_x$ be the open connected component of G that contains x. $\Rightarrow$ $\mathring{I}_x$ is an open interval and if $x, y \in G$ then either $\mathring{I}_x \cap \mathring{I}_y = \phi$ or $\mathring{I}_x = \mathring{I}_y$. $\Rightarrow \mathcal{C} = \{ \mathring{I}_x \mid x \in G \cap \mathbb{Q} \}$ is the desired cover.

2) First let

$$Q_l = [0, 2^{-l}]^n$$ be the dyadic cube w/ side length 2^{-l}.

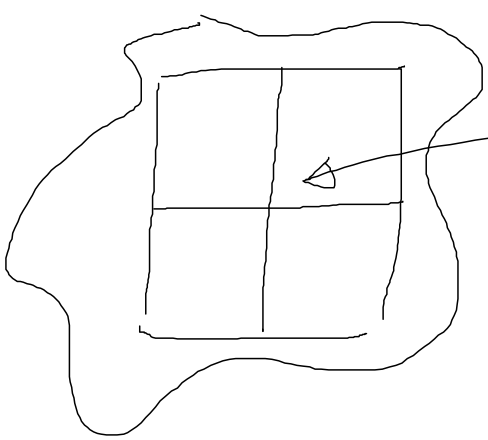
let $$K_l = \left\{ \frac{\vec{k}}{2^l} + Q_l \;\middle|\; \vec{k} \in \mathbb{Z}^n \right\}$$ be the translates of Q_l so that the corners are on the dyadic lattice in $\mathbb{R}^n$.

Note that $l \leq m$ and $Q \in K_l$ $Q' \in K_m$ then either $Q' \subseteq Q$ or $(Q')^\circ \cap Q^\circ = \phi$.



Fix $G \subseteq \mathbb{R}^n$ open and $\delta > 0$.
Take n_0 to be the smallest natural number so that
$$2^{-n_0}\sqrt{n} < \delta$$
(recall that $\operatorname{diam}(Q_{n_0}) = \sqrt{n}\, 2^{-n_0}$ since $Q_{n_0} \subseteq \mathbb{R}^n$)

Now, let $\mathcal{C}_{n_0} = \{Q \in K_{n_0} \mid Q \subseteq G\}$. i.e. we create a nonoverlapping cover of diam $< \delta$ that is contained in G. Note: we want $G = \bigcup$ cubes not $G \supseteq \bigcup$ cubes.



doesn't fill up G. Need to add on some smaller cubes.

Given $\mathcal{C}_k$, we use it to define $\mathcal{C}_{k+1}$ as follows
$$\mathcal{C}_{k+1} = \{Q' \in K_{k+1} \mid Q' \subseteq G \text{ and } (Q')^\circ \cap Q^\circ = \emptyset \text{ for every } Q \in \bigcup_{n_0}^{k} \mathcal{C}_\ell\}.$$

By induction, we create a countable family of nonoverlapping cubes of diam $< \delta$. We want
$$G = \bigcup_{Q \in \mathcal{C}} Q \quad \text{where } \mathcal{C} = \bigcup_{n_0}^{\infty} \mathcal{C}_k$$

Now, if $m \le l$, $Q \in \mathcal{C}_m$ and $Q' \in \mathcal{C}_l$

then either $Q = Q'$ or $Q^\circ \cap (Q')^\circ = \emptyset$.

$\Rightarrow$ $\mathcal{C} = \bigcup_{n_1}^{\infty} \mathcal{C}_k$ is nonoverlapping countable, and

$\bigcup_{Q \in \mathcal{C}} Q \subseteq G$. Let $x \in G$. Choose $m \ge n_0$ and

$Q' \in K_m$ so that $x \in Q' \subseteq G$. (can do this

since l^∞-metric equivalent to l^2-metric)

If $Q' \notin \mathcal{C}_m$ then $\exists$ l $n_0 \le l < m$ so that

$Q' \subseteq Q$ for some $Q \in \mathcal{C}_l$ (since we know

that $(Q')^\circ \cap (Q)^\circ \neq \emptyset$.).

$$\Rightarrow \quad x \in Q' \subseteq Q \subseteq \bigcup_{Q \in \mathcal{C}} Q$$

this shows $G \subseteq \bigcup_{Q \in \mathcal{C}} Q$ $\therefore$ $G = \bigcup_{Q \in \mathcal{C}} Q$ as desired. //

<u>lemma 2.1.10</u> If Γ is Lebesgue measurable then so is Γ^c.

If you recall, we defined Lebesgue measurability via an outer measure and proved things like $\exists\, G \in \mathcal{G}_\delta$ so that

$$G \subseteq \Gamma \quad \text{and} \quad |G|_e = |\Gamma|_e$$

and then ~~we~~ had a theorem that said

$$|G|_e = 0 \Rightarrow G\text{ - Lebesgue measurable.}$$

You can equally well imagine defining an inner measure via approximations of Γ from within by closed sets and then getting results concerning F_σ.

If Lebesgue measure is consistent with respect to inner & outer measures then it should be closed under complements.

The proof of lemma 2.1.10 is (basically) doing this but w/o defining inner measures.

proof:

First, check that K compact $\Rightarrow$ K Lebesgue measurable.

Fix $\varepsilon > 0$ and choose an open set G, $K \subseteq G$ so that $|G|_e \leq |K|_e + \varepsilon$. Since G is L. measurable,

$|G|_e = |G| \Rightarrow |G| \leq |K|_e + \varepsilon$. Now take $H = G - K$.

H is open $\Rightarrow$ by lemma 2.1.9. $\exists$ a countable nonoverlapping cover by cubes so that $H = \bigcup_1^\infty Q_k$.

For any finite subcollection, by lemma 2.1.1

$$\sum_1^m \mathrm{vol}(Q_k) \overset{\text{lemma 2.1.7}}{=} \sum_1^m |Q_k|$$

$$= \left|\bigcup_1^m Q_k\right|_e = \left|\bigcup_1^m Q_k\right|$$

lemma 2.1.1 (first $=$), lemma 2.1.7 (second $=$)

Since K and $\sum_1^m Q_k$ are disjoint compact sets,

$\mathrm{dist}(K, \sum_1^m Q_k) > 0. \Rightarrow |K \cup (\bigcup_1^m Q_k)|_e = |\bigcup_1^m Q_k| + |K|_e$

$$\Rightarrow \quad |G| = |G|_e \geq \left|K \cup (\bigcup_1^m Q_k)\right|_e$$

$$= \left|\bigcup_1^m Q_k\right| + |K|_e = \sum_1^m |Q_k| + |K|_e$$

$\therefore \sum_1^m |Q_k| \leq |G| - |K|_e < \varepsilon$ true $\forall m$.

$\Rightarrow \sum_1^\infty |Q_k| \leq \varepsilon. \Rightarrow |G - K|_e \leq \sum_1^\infty |Q_k| \leq \varepsilon$

$\Rightarrow$ K is Lebesgue measurable.

Now that we've shown that compact sets are Lebesgue measurable, we show that closed sets are Lebesgue measurable

Let F be a closed set.

$$\Rightarrow F = \bigcup_{1}^{\infty} \left(F \cap [B(0,k)]\right)$$

where $B(0,k)$ is the open ball of radius k, centered at 0.

$[B(0,k)]$ is L. measurable since compact.

$F \cap [B(0,k)]$ is a closed subset of a compact set hence compact hence Lebesgue measurable.

By lemma 2.1.7, the countable union of Lebesgue measurable sets is Lebesgue measurable

$\Rightarrow F$ is Lebesgue measurable.

Now closed sets being Lebesgue measurable + lemma 2.1.7 $\Rightarrow$ unions of closed sets are Lebesgue measurable

$\Rightarrow \mathcal{F}_\sigma \subseteq \overline{\mathcal{B}}_{\mathbb{R}^n}$. Now we're ready!

Let Γ be L. measurable $\Rightarrow \exists\, B \in \mathcal{G}_\delta$ s.t. that $\Gamma \subseteq B$ and $|B - \Gamma|_e = 0$. $B^c \in \mathcal{F}_\sigma \Rightarrow B^c$ measurable,

$\Rightarrow \Gamma^c = B^c \cup \{B - \Gamma\}$ is Lebesgue measurable.

$$\Gamma^c = B^c \cup \{B - \Gamma\}$$

L. measurable since $B^c \in \mathcal{F}_\sigma$

L. measurable since $|B - \Gamma|_e = 0 \Rightarrow B - \Gamma$ is L. meas.

$\Rightarrow \Gamma^c$ is the finite union of Lebesgue measurable sets hence it's Lebesgue measurable. //

Now we're nearly done w/ our construction.

We want

1) a good descr. of $\overline{\mathcal{B}}_{\mathbb{R}^n}$

2) to show that Lebesgue measure is countably additive

3) to show that it behaves properly under linear transformations.

4) to see if $\exists$ sets that are not Lebesgue measurable.

Before doing this we need one last lemma.

<u>Lemma</u>: if $a_{mn} \geq 0$ then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn}$$

proof:
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} := \lim_{M\to\infty}\lim_{N\to\infty}\sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn} := \lim_{N\to\infty}\lim_{N\to\infty}\sum_{n=1}^{N}\sum_{m=1}^{M} a_{mn}$$

since $a_{mn} \geq 0$ $\forall m,n$ we know the partial sums are increasing and bounded above by the series.

$$\therefore \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \geq \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}$$
$$= \sum_{n=1}^{N}\sum_{m=1}^{M} a_{mn}$$

$\therefore \sum_{m}\sum_{n} a_{mn}$ is an upper bound for $\sum_{n=1}^{N}\sum_{m=1}^{M} a_{mn}$

$\therefore$ take $M\to\infty$. then take $N\to\infty$.

$$\Rightarrow \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \geq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}.$$

Similarly, $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn} \geq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}$

$\Rightarrow$ if the series are finite then they are equal. //

<u>Theorem</u>: the class $\overline{\mathcal{B}_{\mathbb{R}^n}}$ contains $\mathcal{G}$, is closed under countable unions, complementation and therefore also under differences and countable intersections. Hence $\mathcal{G}_\delta \cup \mathcal{F}_\sigma \subseteq \overline{\mathcal{B}_{\mathbb{R}^n}}$. In fact $\Gamma \in \overline{\mathcal{B}_{\mathbb{R}^n}}$ if and only if $\exists$ $A \in \mathcal{F}_\sigma$ and $B \in \mathcal{G}_\delta$ so that $A \subseteq \Gamma \subseteq B$ and $|A - B| = 0$ (implying $|A| = |\Gamma| = |B|$). Finally, for any $\{\Gamma_n\}_1^\infty \subseteq \overline{\mathcal{B}_{\mathbb{R}^n}}$ with $\Gamma_m \cap \Gamma_n = \phi$ $\forall m, n$

$$\left| \bigcup_1^\infty \Gamma_k \right| = \sum_{k=1}^\infty |\Gamma_k| .$$

<u>proof</u>: We've already proven everything up to and including "$\mathcal{G}_\delta \cup \mathcal{F}_\sigma \subseteq \overline{\mathcal{B}_{\mathbb{R}^n}}$". Now let Γ be in $\overline{\mathcal{B}_{\mathbb{R}^n}}$. We want to find $A \subseteq \Gamma \subseteq B$ w/ $A \in \mathcal{F}_\sigma$, $B \in \mathcal{G}_\delta$ and $|A - B| = 0$.

1) Since Γ is measurable, $\exists$ $G_n \in \mathcal{G}$ so that $|G_n - \Gamma|_e < \frac{1}{n}$ $\Rightarrow$ $B = \bigcap_1^\infty G_n$ satisfies $\Gamma \subseteq B$ $B \in \mathcal{G}_\delta$ and $|B - \Gamma|_e = 0$ $\Rightarrow$ $|B - \Gamma| = 0$.

Since Γ^c is measurable, $\exists\ \tilde{A} \in \mathcal{G}_\delta$ so that $\Gamma^c \subseteq \tilde{A}$ and $|\tilde{A} - \Gamma^c| = 0$.

Take $A = \tilde{A}^c$. Then $A \in \mathcal{F}_\sigma$ and $A \subseteq \Gamma$

and $\tilde{A} - \Gamma^c = \Gamma - \tilde{A}^c = \Gamma - A \Rightarrow |\Gamma - A| = 0$.

Finally, $|B - A| = |(B - \Gamma) \cup (\Gamma - A)|$

$\leq |B - \Gamma| + |\Gamma - A| = 0 + 0 \Rightarrow |B - A| = 0$.

2). Assume $\Gamma \subseteq \mathbb{R}^n$ and $\exists\ A \in \mathcal{F}_\sigma$, $B \in \mathcal{G}_\delta$ so that $A \subseteq \Gamma \subseteq B$. and $|B - A| = 0$. we want to show $\Gamma \in \overline{\mathcal{B}}_{\mathbb{R}^n}$.

We know $\Gamma = A \cup (\Gamma - A)$

and $|\Gamma - A|_e \leq |B - A|_e = |B - A| = 0$

$\Rightarrow \Gamma - A$ is Lebesgue measurable since its outer measure is 0.

$\therefore\ \Gamma$ is the union of two Lebesgue measurable sets

$\Rightarrow$ it's in $\overline{\mathcal{B}}_{\mathbb{R}^n}$.

$A \subseteq \Gamma \subseteq B \Rightarrow |A| \leq |\Gamma| \leq B$

now $|\Gamma| = |A \cup (\Gamma - A)| \leq |A| + |\Gamma - A| \leq |A| + |B - A| = |A|$

$\Rightarrow |\Gamma| \leq |A| \Rightarrow |\Gamma| = |A|$

similarly, $|B| \leq |\Gamma| + |B - \Gamma| \leq |\Gamma| + |B - A| = |\Gamma|$

$\Rightarrow |B| \leq |\Gamma| \Rightarrow |\Gamma| = |B|$.

3) So we're down to proving the countable summability. $\left|\bigcup_1^\infty \Gamma_k\right| = \sum_1^\infty |\Gamma_k|$ if pairwise disjoint.

First, assume the Γ_k's are all bounded

Fix $\varepsilon > 0$ $(\Gamma_k)^c \in \mathcal{B}_{\mathbb{R}^n} \Rightarrow \exists\, G_k$ open so that

$$(\Gamma_k)^c \subseteq G_k \quad \text{and} \quad |G_k - (\Gamma_k)^c| < \frac{\varepsilon}{2^k}.$$

then $G_k - (\Gamma_k)^c = \Gamma_k - \underbrace{(G_k)^c}$ closed and bounded (since contained in Γ_k) hence compact.

$\Rightarrow$ we've found $K_k \subseteq \Gamma_k$ w/ K_k compact and

$$|\Gamma_k - K_k| < \varepsilon/2^k \Rightarrow |\Gamma_k| \leq |K_k| + \varepsilon/2^k$$

Since $\Gamma_k \cap \Gamma_l = \emptyset$ for $k \neq l$, this implies $K_k \cap K_l = \emptyset$ for $k \neq l$.

$\Rightarrow \operatorname{dist}(K_k, K_l) > 0$ (since both sets compact)

$$\Rightarrow \quad \left|\bigcup_1^n K_k\right| = \sum_1^n |K_k| \quad \text{by lemma 2.1.2}$$

$$\therefore \sum_1^\infty |\Gamma_k| \leq \sum_1^\infty |K_k| + \varepsilon/2^k = \varepsilon + \sum_1^\infty |K_k|$$

$$= \varepsilon + \lim_{N\to\infty} \sum_1^N |K_k| = \varepsilon + \lim_{N\to\infty} \left|\bigcup_1^N K_k\right|$$

$$\leq \varepsilon + \left|\bigcup_1^\infty \Gamma_k\right|.$$

$$\Rightarrow \sum_1^\infty |\Gamma_k| \le \varepsilon + \left|\bigcup_1^\infty \Gamma_k\right|$$

true $\forall \varepsilon \Rightarrow$ $\sum_1^\infty |\Gamma_k| \le \left|\bigcup_1^\infty \Gamma_k\right|$.

and since $\left|\bigcup_1^\infty \Gamma_k\right| \le \sum_1^\infty |\Gamma_k|$ automatically, we're done in the case where each Γ_k is bounded.

Now for the general case.

$$A_1 := B(0,1) \qquad A_{n+1} := B(0,n+1) - B(0,n)$$

$\Rightarrow (\Gamma_m \cap A_n) \cap (\Gamma_{m'} \cap A_{n'}) = \emptyset$ if $(m,n) \ne (m',n')$.

and each $\Gamma_m \cap A_n$ is a bounded set.

$$\Gamma_m = \bigcup_1^\infty (\Gamma_m \cap A_n)$$

by previous part of this proof, $|\Gamma_m| = \sum_{n=1}^\infty |\Gamma_m \cup A_n|$

$$\Rightarrow \sum_{m=1}^\infty |\Gamma_m| = \sum_{m=1}^\infty \sum_{n=1}^\infty |\Gamma_m \cap A_n| = \sum_{n=1}^\infty \sum_{m=1}^\infty |A_n \cap \Gamma_m|$$

$$= \sum_{n=1}^\infty \left| \bigcup_{m=1}^\infty (A_n \cap \Gamma_m) \right| = \sum_{n=1}^\infty \left| A_n \cap \left(\bigcup_1^\infty \Gamma_m\right) \right|$$

$$= \left| \bigcup_{n=1}^\infty \left[A_n \cap \left(\bigcup_{m=1}^\infty \Gamma_m \right) \right] \right| = \left| \bigcup_{m=1}^\infty \Gamma_m \right|.$$ done!

That finishes the construction of Lebesgue measure. Now you want to know

1) does it behave well wrt linear transformations

2) are there sets that aren't Lebesgue measurable?

See pp 26-27 of Stroock for the construction of non-Lebesgue-measurable sets.

Now we want to show that if

$T : \mathbb{R}^n \to \mathbb{R}^n$ is linear

and $\Gamma \in \overline{\mathcal{B}}_{\mathbb{R}^n}$ then $|T\Gamma| = |\det(T)|\,|\Gamma|$

This will show that our measure is rotation invariant and that it respects dilation as expected: $|\alpha\Gamma| = |\alpha|^n\,|\Gamma|$

<u>Note</u>: the measure is translation invariant since the volume of a rectangle is translation invariant

Intuitively, the Lebesgue measure is built on rectangles and since $\mathrm{vol}(TI) = |\det(T)|\,\mathrm{vol}(I)$ for a rectangle all should be well.

First, prove a lemma about general continuous mappings and then we'll restrict to linear mappings.

<u>lemma 2.2.1:</u> Let $F \subseteq \mathbb{R}^M$ be a closed set and $\phi : F \to \mathbb{R}^N$ be continuous.

1) if $\Gamma \in \mathcal{F}_\sigma$ then $\phi(F \cap \Gamma) \in \mathcal{F}_\sigma$

2) if $|\Gamma|_e = 0 \Rightarrow |\phi(\Gamma \cap F)|_e = 0$ for all $\Gamma \subseteq \mathbb{R}^M$ w/ $|\Gamma|_e = 0$ then $\Gamma \in \overline{\mathcal{B}_{\mathbb{R}^M}} \Rightarrow \phi(\Gamma \cap F) \in \overline{\mathcal{B}_{\mathbb{R}^N}}$.

3) if $M \leq N$ and ϕ is Lipschitz continuous then $|\phi(\Gamma \cap F)|_e \leq (2\sqrt{N}L)^N |\Gamma|_e$ where L is the Lipschitz constant of L. Hence ϕ takes measurable subsets of F to measurable subsets of $\mathbb{R}^N$.

<u>Q:</u> What's with all the intersecting w/ F? Well, to some degree, it allows us to define these things when ϕ isn't defined on all $\mathbb{R}^M$. Or ϕ might not be Lipschitz on all $\mathbb{R}^M$.

recall $\phi\left(\bigcup_\alpha V_\alpha\right) = \bigcup_\alpha \phi(V_\alpha)$ if ϕ is a function.

<u>proof of lemma:</u>

1) Let $\Gamma \in \mathcal{F}_\sigma \Rightarrow \Gamma = \bigcup_1^\infty F_n$ where F_n is closed.

$\Rightarrow \phi(F \cap \Gamma) = \bigcup_1^\infty \phi(F_n \cap F)$.

we know that if K is compact then $K \cap F$ is compact and since ϕ is continuous, we know $\phi(K \cap F)$ is compact. Now, any closed set in $\mathbb{R}^m$ is the countable union of compact sets.

$$\Rightarrow \quad F_k \cap F = \left(\bigcup_1^\infty K_n\right) \cap F = \bigcup_1^\infty (K_n \cap F)$$

$$\Rightarrow \quad \phi(F_k \cap F) = \bigcup_1^\infty \underbrace{\phi(K_n \cap F)}_{\text{compact, hence closed.}}$$

$\Rightarrow \phi(F_k \cap F) \in \mathcal{F}_\sigma$. And now, since $\mathcal{F}_\sigma$ is closed under countable unions, $\phi(\Gamma \cap F) = \bigcup_1^\infty \phi(F_k \cap F) \in \mathcal{F}_\sigma$. ✓

2) assume that $|\Gamma|_e = 0 \Rightarrow |\phi(\Gamma \cap F)|_e = 0$. We want to show $\Gamma \in \overline{\mathcal{B}_{\mathbb{R}^m}} \Rightarrow \phi(\Gamma \cap F) \in \overline{\mathcal{B}_{\mathbb{R}^n}}$

Since Γ is Lebesgue measurable, $\exists\, A \in \mathcal{F}_\sigma$ so that $A \subseteq \Gamma$ and $|\Gamma - A| = 0$. $\Gamma \cap F = (A \cap F) \cup ((\Gamma - A) \cap F)$

$$\rightarrow \phi(\Gamma \cap F) = \underbrace{\phi(A \cap F)}_{\text{is } \mathcal{F}_\sigma \text{ by part 1) hence is Lebesgue measurable.}} \cup \phi((\Gamma - A) \cap F)$$

and $|\phi((\Gamma - A) \cap F)|_e = 0$

since $|(\Gamma - A) \cap F|_e \leq |\Gamma - A|_e = 0$.

$\phi((\Gamma - A) \cap F)$ is Lebesgue measurable because its outer measure $= 0$. $\Rightarrow \phi(\Gamma \cap F)$ is Lebesgue measurable.

3) Assume ϕ is Lipschitz continuous w/ Lipschitz constant L. We want to show

$$\circledast \quad |\phi(\Gamma \cap F)|_e \le (2\sqrt{N}\,L)^N\,|\Gamma|_e \quad \forall\, \Gamma \subseteq \mathbb{R}^M.$$

(this will then imply that if ϕ is Lipschitz then ϕ takes measurable sets to measurable sets since $\circledast$ implies $|\Gamma|_e = 0 \Rightarrow |\phi(\Gamma \cap F)|_e = 0$ and we're done by part 2).)

Assume we know $\circledast$ holds if Γ is a cube Q w/ diam < 1.

Let Γ be an arb. subset of $\mathbb{R}^M$ with $|\Gamma|_e < \infty$. then $\exists$ open set $G \subseteq \mathbb{R}^n$ with $|G|_e \le |\Gamma|_e + \varepsilon$. by lemma 2.1.9 we can find a countable collection of nonoverlapping cubes $\mathcal{C}$ each of which has diam < 1 so that

$$G = \bigcup_{Q \in \mathcal{C}} Q$$

And since cubes are measurable and we've shown that the Lebesgue measure is countably additive for a family of mutually disjoint measurable sets,

$$|G| = \Big|\bigcup_{Q \in \mathcal{C}} Q\Big| = \sum_{Q \in \mathcal{C}} |Q| \le |\Gamma|_e + \varepsilon$$

$$\Rightarrow \left|\phi(\Gamma\cap F)\right|_e \le \left|\phi(G\cap F)\right|_e = \left|\bigcup_{Q\in\mathcal{C}} \phi(Q\cap F)\right|_e$$

$$\le \sum_{Q\in\mathcal{C}} \left|\phi(Q\cap F)\right|_e$$

$$\le \sum_{Q\in\mathcal{C}} (2\sqrt{N}L)^N |Q|_e$$ since diam $Q<1$ and we assumed ⊛ holds for such cubes

$$= (2\sqrt{N}L)^N \sum_{Q\in\mathcal{C}} |Q|_e$$

$$= (2\sqrt{N}L)^N \sum_{Q\in\mathcal{C}} |Q|$$ since $|Q|_e = |Q|$

$$\le (2\sqrt{N}L)^N \left[|\Gamma|_e + \varepsilon\right]$$

This is true $\forall \varepsilon>0 \Rightarrow |\phi(\Gamma\cap F)|_e \le (2\sqrt{N}L)^N |\Gamma|_e$.

So it remains to prove that ⊛ holds for cubes of diam D, $D<1$. Note: if $Q\cap F=\phi$ then ⊛ is trivially true. So assume $Q\cap F\ne\phi$. Let $x\in Q\cap F$. Then $\phi(x)\in\phi(Q\cap F)$.

Since ϕ is Lipschitz, we know that $|x-y|\le D$

$\Rightarrow |\phi(x)-\phi(y)| \le L|x-y| \le LD \Rightarrow \phi(Q\cap F)$ is contained in a ball of radius LD centered at $\phi(x)$.

$\Rightarrow \phi(Q\cap F)$ is contained in the cube

$$J = \prod_1^N \left[(\phi(x))_k - LD, (\phi(x))_k + LD\right]$$

$\Rightarrow \quad |\phi(Q \cap F)|_e \leq |J|_e = (2LD)^N$

Now the cube Q has side length q

$\Rightarrow$ volume $(Q) = |Q| = q^M$ $\quad D = \operatorname{diam}(Q) = \sqrt{N}\, q$

$\Rightarrow \quad D = \sqrt{N}\, |Q|^{\frac{1}{M}}$

$\Rightarrow |\phi(Q \cap F)|_e \leq (2L)^N (\sqrt{N})^N |Q|^{\frac{N}{M}} = (2L\sqrt{N})^N |Q|^{\frac{N}{M}}$.

We assumed $N \geq M$ and $|Q| \leq 1 \Rightarrow |Q|^{\frac{N}{M}} \leq |Q|$

$\therefore \quad |\phi(Q \cap F)|_e \leq (2L\sqrt{N})^N |Q|$ as desired. //

Okay, now we're ready to prove our big theorem about Lebesgue measure behaving well under linear transformations.

Because we committed already to a set of coordinates for $\mathbb{R}^N$ when we defined the Lebesgue measure, we won't talk about an arbitrary linear transformation. Instead, we'll work with the matrix A that our coordinates give us for the linear transformation

I.e. if A is an $N \times N$ matrix then $T_A : \mathbb{R}^N \to \mathbb{R}^N$ is a linear transformation on $\mathbb{R}^N$ given by

$$T_A(\vec{x}) := A\vec{x}$$

Theorem: Given a real $N \times N$ matrix A, T_A takes L. measurable sets to L. measurable sets and $|T_A(\Gamma)|_e \le |\det(A)|\,|\Gamma|_e$ for all subsets $\Gamma \subseteq \mathbb{R}^N$.

Proof: Since T_A is Lipschitz, lemma 2.2.1 takes care of the first part

Step 1: First show that if $\vec{c} \in \mathbb{R}^N$, $\lambda \in \mathbb{R}$, and $\Gamma \subseteq \mathbb{R}^N$ then $|\vec{c} + \lambda\Gamma|_e = |\lambda|^N |\Gamma|_e$ where $\lambda\Gamma = \{\lambda\vec{x} \mid \vec{x} \in \Gamma\}$

Because the Lebesgue measure is translation invariant, WLOG, assume $\vec{c} = \vec{0}$. Also, assume $\lambda \neq 0$ since it's trivially true if $\lambda = 0$. Also, it's clear that if I is a rectangle then $|\lambda I| = |\lambda|^N |I|$ (since $|I| = \mathrm{vol}(I)$.)

recall, $|\Gamma|_e = \inf\limits_{\mathcal{C}} \{ \Sigma(\mathcal{C}) \mid \mathcal{C}$ is a countable cover by rectangles of $\Gamma \}$

and if $\mathcal{C}$ is a cover then $\lambda\mathcal{C}$ is a cover of $\lambda\Gamma$ where $\lambda\mathcal{C} = \{\lambda I \mid I \in \mathcal{C}\}$. Since $|\lambda I| = |\lambda|^N |I|$, we see that $\Sigma(\lambda\mathcal{C}) = |\lambda|^N \Sigma(\mathcal{C})$. And since $\mathcal{C}$ is a cover of $\Gamma \Leftrightarrow \lambda\mathcal{C}$ is a cover of $\lambda\Gamma$ we have

$$\inf_{\mathcal{C}} \{\Sigma(\mathcal{C}) \mid \mathcal{C} \text{ covers } \lambda\Gamma\}$$
$$= \inf_{\mathcal{C}} \{\Sigma(\lambda\mathcal{C}) \mid \mathcal{C} \text{ covers } \Gamma\}$$
$$= |\lambda|^N \inf_{\mathcal{C}} \{\Sigma(\mathcal{C}) \mid \mathcal{C} \text{ covers } \Gamma\}$$

$$\Rightarrow \quad |\lambda\Gamma|_e = |\lambda|^N |\Gamma|_e.$$

Step 2: Now show that if T is a linear transformation and $Q_0 = [0,1]^N$ then $|T(Q)| = \alpha(T)|Q|$ for any cube Q where $\alpha(T) = |T(Q_0)|$.

Let Q be a cube. Then $Q = \vec{c} + \lambda Q_0$ for some $\vec{c}$ and λ. where $|\lambda|^N = |Q|$. $TQ = T\vec{c} + \lambda TQ_0$ since T is linear

$\Rightarrow |TQ| = |T\vec{c} + \lambda TQ_0| = |\lambda TQ_0| = |\lambda|^N |TQ_0| = |TQ_0||Q|$

$= \alpha(T)|Q|$ by defn of $\alpha(T)$. ✓

Step 3: Show that if G is an open set and T is linear then $|T(G)| \leq \alpha(T)|G|$ with equality holding if T is nonsingular.

Let $G \subseteq \mathbb{R}^N$ be open by Lemma 2.1.9 $\exists\ \mathcal{C}$ a countable cover by nonoverlapping cubes so that

$G = \bigcup_{Q \in \mathcal{C}} Q$

$\Rightarrow |TG| = |\bigcup_{Q \in \mathcal{C}} TQ| \leq \sum_{Q \in \mathcal{C}} |TQ| = \sum \alpha(T)|Q| = \alpha(T)|G|$

Since $|G| = \sum_{Q \in \mathcal{C}} |Q|$. $\Rightarrow |TG| \leq \alpha(T)|G|$ ✓.

Note: we know that if $G = \bigcup_1^\infty Q_j$ where $Q_j \cap Q_k = \emptyset$ then $|G| = \sum_1^\infty |Q_j|$ This can be strengthened: if $G = \bigcup_1^\infty Q_j$ where $|Q_j \cap Q_k| = 0$ (intersection has zero measure) then

$|G| = \sum_1^\infty |Q_j|$. We used this above.

We know $G = \sum_1^\infty Q_k \Rightarrow TG = \bigcup_1^\infty TQ_u$. these are not disjoint but they do have disjoint interiors. $\Rightarrow$

$$|TG| = \left|\bigcup_1^\infty TQ_u\right| \geq \left|\bigcup_1^\infty (TQ_u)^\circ\right| = \sum_1^\infty |(TQ_u)^\circ|$$

if we know $|(TQ)^\circ| = |TQ|$ then we'd be done!

Well $|TQ| = |(TQ)^\circ \cup \partial(TQ)| = |(TQ)^\circ| + |\partial(TQ)|$

So if $|\partial(TQ)| = 0$ then we're done.

$$\partial(TQ) = [TQ] - (TQ)^\circ$$

If T is non singular then $[TQ] = TQ$ and

$(TQ)^\circ = T(Q^\circ)$. $\Rightarrow \partial(TQ) = TQ - T(Q^\circ) = T(Q - Q^\circ) = T(\partial Q)$.

$\Rightarrow |\partial(TQ)| = |T(\partial Q)|$. But we know $|\partial Q| = 0$ and

by lemma 2 2.1 we know $T(\partial Q)$ is Lebesgue measurable and $|T(\partial Q)| \leq$ some $\# \cdot |\partial Q| = 0$. $\Rightarrow |\partial(TQ)| = 0$

$\Rightarrow |TQ| = |(TQ)^\circ|$

$\therefore |TG| \geq \sum_1^\infty |TQ_u| = \alpha(T) \sum_1^\infty |Q_u| = \alpha(T)|G|$.

$\therefore$ if T is non singular, $|TG| = \alpha(T)|G|$. ✓

Step 4: Show that if T is a nonsingular linear transformation and $\Gamma \subseteq \mathbb{R}^N$ then

$$|T\Gamma|_e = \alpha(T)\,|\Gamma|_e$$

by lemma 2.1.4, $|\Gamma|_e = \inf\{|G| \mid \Gamma \subseteq G,\ G \text{ open}\}$.

Since T is nonsingular, G open $\iff$ TG open

$$\Rightarrow \quad |T\Gamma|_e = \inf\{|G| \mid T\Gamma \subseteq G,\ G \text{ open}\}$$

$$= \inf\{|T\tilde{G}| \mid T\Gamma \subseteq T\tilde{G},\ \tilde{G} \text{ open}\}$$

$$= \inf\{|T\tilde{G}| \mid \Gamma \subseteq \tilde{G},\ \tilde{G} \text{ open}\}$$

$$= \alpha(T) \inf\{|\tilde{G}| \mid \Gamma \subseteq \tilde{G},\ \tilde{G} \text{ open}\} \quad \text{used step 3}$$

$$= \alpha(T)\,|\Gamma|_e \quad \checkmark$$

Step 5: Show that if S & T are linear transformations and S is non-singular then $\alpha(S\circ T) = \alpha(S)\,\alpha(T)$.

$$\alpha(S\circ T) := (|S\circ T(Q_0)| = |S(T(Q_0))| = \alpha(S)\,|T(Q_0)| \quad \text{step 3}$$

$$= \alpha(S)\,\alpha(T) \quad \text{defn of } \alpha(T.$$

$\checkmark$

Step 6: Show that if A is an orthogonal matrix then $\alpha(T_A) = 1$

Let $\Gamma = B(\vec{0},1)$ = Ball of radius 1, centered at $\vec{0}$.

Then $T_A\Gamma = \Gamma$. but $|T_A\Gamma| = \alpha(T_A)\,|\Gamma| \Rightarrow \alpha(T_A) = 1$.

Step 7: Show that if A is nonsingular and symmetric then $\alpha(T_A) = |\det(A)|$.

First, if A is diagonal then $\alpha(T_A) = |T_A Q_0| = |\lambda_1 \ldots \lambda_N|$
$= |\det(A)|$

where λ_k is the k^{th} diagonal entry

If A is not diagonal, then $\exists$ orthogonal matrix $\mathcal{O}$ so that

$$A = \mathcal{O} D \mathcal{O}^T.$$

where D = diagonal matrix whose entries are the eigenvalues of A. $\Rightarrow$ $\alpha(A) = \alpha(\mathcal{O})\alpha(D)\alpha(\mathcal{O}^T)$ step 5

$= \alpha(D)$ step 6

$= |\det D| = |\det A|$. ✓

Step 8: Show that if A is nonsingular then $\alpha(T_A) = |\det A|$.

Let $B = \sqrt{AA^T}$, then B is nonsingular and symmetric and $\det(B) = \det A$

by step 7, $\alpha(B) = |\det(B)| = |\det A|$. So we want to show $\alpha(A) = \alpha(B)$ and then done

Let $\mathcal{O} = B^{-1}A$ then $\mathcal{O}^T = A^T(B^{-1})^T = A^T B^{-1}$

$\Rightarrow \mathcal{O}\mathcal{O}^T = B^{-1}AA^TB^{-1} = B^{-1}B \cdot B \cdot B^{-1} = I \Rightarrow \mathcal{O}$ is orthogonal

$\Rightarrow B\mathcal{O} = A \Rightarrow \alpha(A) = \alpha(B)\alpha(\mathcal{O}) = \alpha(B)$ ✓

Note: We're now done for the case of nonsingular matrices!

By step 4,

$$|T_A \Gamma|_e = \alpha(T_A)\,|\Gamma|_e$$

by step 8, $\alpha(T_A) = |\det A|$.

Step 9: Show that if A is singular then $\alpha(T_A) = 0$.

Since A is singular, $\operatorname{range}(T_A) \neq \mathbb{R}^N$.

Choose $\vec{y} \in \mathbb{R}^N$ w/ $\|\vec{y}\| = 1$ and $y \perp \operatorname{range}(T_A)$.

Now choose an orthogonal matrix θ so that $\vec{e}_1 = T_\theta \vec{y}$.

$\Rightarrow \vec{e}_1 \perp \operatorname{range}(T_\theta \circ T_A)$

$\therefore$ we can find a rectangle $I \subseteq \mathbb{R}^{N-1}$ so that

$$T_\theta \circ T_A Q_0 \subseteq \{0\} \times I$$

but then $|T_\theta T_A Q_0| \leq |\{0\} \times I| = 0 \Rightarrow$

$\alpha(T_\theta T_A) = 0 \Rightarrow \alpha(T_A) = 0.$ ✓

Now we're done! We've shown $|T_A \Gamma|_e = |\det A|\,|\Gamma|_e$ $\forall$ matrices A $\forall \Gamma \subseteq \mathbb{R}^N$. //