Defn: Let

$$\|A\|_\sigma = \sup\{|\lambda| \mid \lambda \in \text{spectrum}(A)\}.$$

then $\|A\|_\sigma$ is called the spectral radius of A.

recall the two things we proved earlier:

lemma: E a Banach space

1) if $A \in \mathcal{L}(E,E)$ and $\|A\|_{\mathcal{L}(E,E)} < 1$ then $I - A$ is invertible and

$$(I-A)^{-1} = \sum_0^\infty A^n$$

2) if $|\lambda| > \|A\|_{\mathcal{L}(E,E)}$ then $\lambda I - A$ is invertible

and $$(\lambda I - A)^{-1} = \sum_0^\infty \frac{A^n}{\lambda^{n+1}}$$

We use these to prove

theorem: Let E be a Banach space, $A \in \mathcal{L}(E,E)$

1) spectrum (A) is a nonempty compact set with $\|A\|_\sigma \le \|A\|_{\mathcal{L}(E,E)}$

2) Spectrum (A) is closed, the set of regular points $\mathbb{R}(A)$ is open, and

$f: \mathbb{R}(A) \to \mathcal{L}(E,E)$ given by $f(\lambda) = (\lambda I - A)^{-1}$

is analytic.

Proof.

We've already shown that

$$\|A\|_\sigma \le \|A\|_{\mathcal{L}(E,E)}$$

and that $R(A)$ (the regular points) is an open set.
So we know that spectrum(A) is a closed bounded subset of $\mathbb{C}$. $\Rightarrow$ it's compact. All we need to do is

1) show spectrum$(A) \neq \emptyset$
2) show $f : \lambda \to (\lambda I - A)^{-1}$ is analytic on $R(A)$.

Let $\lambda_0 \in R(A)$. We want to find $\{B_n\}_0^\infty \subseteq \mathcal{L}(E,E)$ and $\varepsilon > 0$ so that $|\lambda - \lambda_0| < \varepsilon \Rightarrow \lambda \in R(A)$

and $\quad f(\lambda) = (\lambda I - A)^{-1} = \sum_0^\infty (\lambda - \lambda_0)^n B_n$

We know $(\lambda I - A) = (\lambda_0 I - A)\left[I - (\lambda_0 - \lambda)(\lambda_0 I - A)^{-1}\right]$

We know $\lambda_0 I - A$ is invertible. and therefore $(\lambda I - A)$ will be invertible if

$$|\lambda - \lambda_0| \, \|(\lambda_0 I - A)^{-1}\| < 1. \quad \Rightarrow |\lambda - \lambda_0| < \frac{1}{\|(\lambda_0 I - A)^{-1}\|}$$

$\Rightarrow$ if $\varepsilon = \dfrac{1}{\|(\lambda_0 I - A)^{-1}\|}$ then $|\lambda - \lambda_0| < \varepsilon \Rightarrow \lambda \in R(A)$.

And, I know

$$\left[I-(\lambda_0-\lambda)(\lambda_0 I-A)^{-1}\right]^{-1} = \sum_0^\infty (\lambda_0-\lambda)^n \left[(\lambda_0 I-A)^{-1}\right]^n$$

$$\Rightarrow (\lambda I-A)^{-1} = \left[\sum_0^\infty (-1)^n(\lambda-\lambda_0)^n\left[(\lambda_0 I-A)^{-1}\right]^n\right](\lambda_0 I-A)^{-1}$$

$$= \sum_0^\infty (\lambda-\lambda_0)^n B_n$$

where $B_n = (-1)^n\left[(\lambda_0 I-A)^{-1}\right]^{n+1}$

We ~~~~ need to check that this sum is absolutely convergent. Since then we'll have shown that $f: \lambda \to (\lambda I-A)^{-1}$ is analytic on $R(A)$.

$$\sum_0^\infty \|(\lambda-\lambda_0)^n B_n\| = \sum_0^\infty |\lambda-\lambda_0|^n \left\|\left[(\lambda_0 I-A)^{-1}\right]^{(n+1)}\right\|$$

$$\leq \sum_0^\infty |\lambda-\lambda_0|^n \|(\lambda_0 I-A)^{-1}\|^{n+1}$$

$$= \|(\lambda_0 I-A)^{-1}\| \sum_0^\infty \Big(\underbrace{|\lambda-\lambda_0|\,\|(\lambda_0 I-A)^{-1}\|}_{<1}\Big)^n$$

$\Rightarrow$ converges. ✓

Now all we need to do is show $\text{Spectrum}(A) \neq \emptyset$.

assume spectrum$(A) = \phi$.

then $f(\lambda) = (\lambda I - A)^{-1}$ is analytic on $\mathbb{C}$.

I claim f is bounded on $\mathbb{C}$

First, since f is continuous, f is bounded on the closed ball $\{\lambda \mid |\lambda| \le 2\|A\|\}$ On the other hand, if $|\lambda| > 2\|A\|$ then

$$(\lambda I - A)^{-1} = \sum_0^\infty \frac{A^n}{\lambda^{n+1}}$$

$$\Rightarrow \|(\lambda I - A)^{-1}\| \le \sum_0^\infty \left\|\frac{A^n}{\lambda^{n+1}}\right\| \le \sum_0^\infty \frac{1}{|\lambda|^{n+1}} \|A\|^n$$

$$= \frac{1}{|\lambda|} \sum_0^\infty \left(\frac{\|A\|}{|\lambda|}\right)^n \le \frac{1}{|\lambda|} \sum_0^\infty \left(\frac{1}{2}\right)^n = \frac{2}{|\lambda|} \le \frac{2}{2\|A\|}$$

$\Rightarrow$ we have a uniform upper bound for $\|f(\lambda)\|$ if $|\lambda| > 2\|A\|$.

$\Rightarrow f$ is bounded & entire $\Rightarrow f$ is constant

$\Rightarrow \lambda I - A = \mu I - A$ for $\lambda \ne \mu$. X.

$\Rightarrow$ spectrum$(A) \ne \phi$ //

<u>theorem</u>. (Spectral Radius Formula)

if E is a Banach space, $A \in \mathcal{L}(E,E)$ then

$$\|A\|_\sigma = \lim_{n\to\infty} \|A^n\|^{1/n}$$

<u>Note</u>: if $E = \mathbb{R}^2$ and A is diagonalizable

then $A = B\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}B^{-1}$

$$\Rightarrow A^n = B\begin{pmatrix}\lambda_1^n & 0\\ 0 & \lambda_2^n\end{pmatrix}B^{-1}$$

by theorem

$$\Rightarrow \|A^n\|^{1/n} \to \underbrace{\max\{|\lambda_1|, |\lambda_2|\}}_{\|A\|_{\mathcal{L}(E,E)}} = \|A\|_\sigma \checkmark$$

$$\Rightarrow \|A\|_\sigma = \|A\|_{\mathcal{L}(E,E)} \text{ in this case.}$$

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if $A = \begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}$ then $AA^* \neq A^*A$

$\sigma(A) = \{1\}$ $\quad \|A\|_\sigma = 1 \quad \|A\|_{\mathcal{L}(E,E)} = \sqrt{1+|\lambda|^2}$

$A^n = \begin{pmatrix}1 & n\lambda\\ 0 & 1\end{pmatrix} \Rightarrow \|A^n\| = \sqrt{1+|n\lambda|^2}$

$$\Rightarrow \|A^n\|^{1/n} \to 1 \quad (= \|A\|_\sigma).$$

<u>proof of spectral radius theorem:</u>

we know spectrum(A) is closed & bounded.

$\Rightarrow$ $(R(A) - \{0\})^{-1} \cup \{0\}$ is open in $\mathbb{C}$.

(first, since $\{|z| > \|A\|\} \subseteq R(A)$, we know

$$R(A)^{-1} \supseteq \{ w \mid w \neq 0 \ |w| < \tfrac{1}{\|A\|} \}.$$

$\Rightarrow$ we fill the hole by adding $\{0\}$.

(check the set is open using tradition δ-Ball arguments.)

we define a function on this set by

$$h(\mu) = \begin{cases} (\frac{1}{\mu} I - A)^{-1} & \text{if } \mu \neq 0 \\ 0 & \text{if } \mu = 0 \end{cases}$$

if μ_0 satisfies $0 < |\mu_0| < \frac{1}{\|A\|_\sigma}$

then $\frac{1}{\mu_0}$ satisfies $\|A\|_\sigma < \frac{1}{\mu_0}$.

$\Rightarrow$ $\exists\, r > 0$ and $\{B_n\} \subseteq \mathcal{L}(E,E)$ so that

abs. conv. series

$$\left|\tfrac{1}{\mu} - \tfrac{1}{\mu_0}\right| < r \Rightarrow \left(\tfrac{1}{\mu} I - A\right)^{-1} = \sum_0^\infty \left(\tfrac{1}{\mu} - \tfrac{1}{\mu_0}\right)^n B_n$$

$$\Rightarrow \left|\tfrac{1}{\mu} - \tfrac{1}{\mu_0}\right| < r \Rightarrow \left(\tfrac{1}{\mu} I - A\right)^{-1} = \sum_0^\infty (\mu - \mu_0)^n C_n$$ $\leftarrow$ abs conv. series.

where $C_n = (-1)^n \dfrac{B_n}{(\mu\mu_0)^n}$

$\Rightarrow$ if μ_0 satisfies $0 < |\mu_0| < \frac{1}{\|A\|_\sigma}$

then $\exists\ \varepsilon > 0$ so that $|\mu - \mu_0| < \varepsilon \Rightarrow$

$$\left(\frac{1}{\mu} I - A\right)^{-1} = \sum_0^\infty (\mu - \mu_0)^n C_n \quad \leftarrow \text{abs conv.}$$

$\therefore$ we've shown h is analytic on $\{ z \mid 0 < |z| < \frac{1}{\|A\|_\sigma} \}$

Also, we know that $|\lambda| \geq \|A\|$

$$\Rightarrow (\lambda I - A)^{-1} = \sum_0^\infty \frac{A^n}{\lambda^{n+1}} \quad \leftarrow \text{abs. conv.}$$

$$\Rightarrow |\lambda| \leq \frac{1}{\|A\|} \Rightarrow \left(\frac{1}{\lambda} I - A\right)^{-1} = \sum_0^\infty \lambda^{n+1} A^n \quad \circledast$$

$\therefore$ $h(\lambda)$ is analytic at 0.

$\Rightarrow$ h is analytic on $\{ z \mid |z| \leq \frac{1}{\|A\|_\sigma} \}$

Furthermore, the expression $\circledast$ holds not only on

$|\lambda| \leq \frac{1}{\|A\|}$ but for the larger domain

$|\lambda| \leq \frac{1}{\|A\|_\sigma}$ since it holds in the largest ball contained in $(R(A) - \{0\})^{-1} \cup \{0\}$.

$\Rightarrow$ radius of convergence, R, of $\circledast$ satisfies

$$R \geq \frac{1}{\|A\|_\sigma}$$

Can $R > \frac{1}{\|A\|_\sigma}$? If this were true then $\lambda I - A$ would be invertible for all $|\lambda| > \frac{1}{R}$.

If $R > \frac{1}{\|A\|_\sigma}$ then $\frac{1}{R} < \|A\|_\sigma$

$\Rightarrow \sup\{|\lambda| \mid \lambda \in \text{spectrum}(A)\} < \|A\|_\sigma$. $\times$

$\Rightarrow$ radius of convergence of $\circledast$ is $\frac{1}{\|A\|_\sigma}$.

$\Rightarrow$ by lemma for analytic functions,

$$R = \left(\overline{\lim}\, \|A^n\|^{1/n}\right)^{-1} = \frac{1}{\|A\|_\sigma}$$

$\Rightarrow \|A\|_\sigma = \overline{\lim}\, \|A^n\|^{1/n}$.

Now to show $\overline{\lim}\, \|A^n\|^{1/n} = \lim \|A^n\|^{1/n}$.

We do this as follows let $\{r_n\}$ be such that $r_n \geq 0$ $\forall n$. and $\overline{\lim\limits_{n\to\infty}}\, r_n < \infty$. then if we can show $r_m \geq \overline{\lim\limits_{n\to\infty}}\, r_n$ for each m then this will imply $\underline{\lim}\, r_n \geq \overline{\lim}\, r_n \Rightarrow \underline{\lim} = \overline{\lim} = \lim$.

Fix $m > 0$. If $n > m$, write

$n = qm + r$ where $q, r \in \mathbb{N}$.

$$\Rightarrow \|A^n\|^{1/n} = \|A^{qm} A^r\|^{\frac{1}{n}} \leq \|A^{qm}\|^{\frac{1}{n}} \|A^r\|^{\frac{1}{n}}$$

$$\leq \|A^m\|^{\frac{q}{n}} \|A^r\|^{1/n}$$

$$= \left(\|A^m\|^{\frac{1}{m}}\right)^{\frac{qm}{n}} \|A^r\|^{1/n}$$

$$\leq \left(\|A^m\|^{\frac{1}{m}}\right)^{1-\frac{r}{n}} \|A\|^{r/n}$$

Since $r < m$, as $n \to \infty$ $\frac{r}{n} \to 0$. $\Rightarrow \|A\|^{\frac{r}{n}} \to 1$

and

$$\overline{\lim_{n\to\infty}} \|A^n\|^{1/n} \leq \overline{\lim_{n\to\infty}} \left(\|A^m\|^{\frac{1}{m}}\right)^{1-\frac{r}{n}} \|A\|^{r/n}$$

$$= \|A^m\|^{\frac{1}{m}}$$

Since m was arbitrary, we're done by previous observation on sequences //

<u>Corollary</u>: if E is a Hilbert Space, $A \in \mathcal{L}(E,E)$ is normal, then $\|A\|_\sigma = \|A\|$

<u>Proof</u>: we know $\|A\|_\sigma = \lim_{n\to\infty} \|A^n\|^{1/n}$

So it suffices to find a sequence of integers so that $\|A^n\| = \|A\|^n$

since then $\|A\|_\sigma = \lim \|A^n\|^{1/n} = \lim_{n\to\infty} (\|A\|^n)^{1/n} = \|A\|.$

I'll show $\|A^2\| = \|A\|^2$ and then done since it will imply $\|A^{2^n}\| = \|A\|^{2^n}$ $\forall n.$

$$\|A\|^2 = \sup_{\|x\|\le 1} \langle Ax, Ax\rangle = \sup_{\|x\|\le 1} |\langle A^*Ax, x\rangle|$$

$$= \|A^*A\|$$

since A^*A self adjoint

$$(A^*A)^* = A^*A^{**} = A^*A.$$

On the other hand

$$\|A^*A\| = \sup_{\|x\|=1} \sqrt{\langle A^*Ax, A^*Ax\rangle}$$

$$= \sup_{\|x\|=1} \sqrt{\langle AA^*Ax, Ax\rangle} = \sup_{\|x\|=1} \sqrt{\langle A^*AAx, Ax\rangle}$$

$$= \sup_{\|x\|=1} \sqrt{\langle AAx, AAx\rangle} = \|A^2\|.$$ done! //