$$(L, \tau) \rightarrow (L^*, b) \rightarrow (L^{**}, b^*)$$

We can use (E, τ) to induce weak$*$ topology τ_{w*} on L^*

<u>thm:</u> $f_n \xrightarrow[w*]{} f_0$ in L^* $\iff$ $f_n(x) \to f_0(x)$ $\forall x \in L$

we can use L^{**} to induce weak topology τ_w on L^*

<u>thm:</u> $f_n \xrightarrow[w]{} f_0$ in L^* $\iff$ $\Psi(f_n) \to \Psi(f_0)$ $\forall \Psi \in L^{**}$

recall, if (L, τ) is reflexive then $\tau_w = \tau_{w*}$.
Otherwise $\tau_{w*} \subseteq \tau_w \subseteq b$.

Last time, discussed weak$*$ boundedness and weak boundedness. Today, we'll consider compactness and relative compactness.

<u>thm:</u> Assume $(L, \|\cdot\|)$ is <u>separable</u>, consider $(L^*, \|\cdot\|)$. If $\{f_n\}$ is (strongly) bounded then $\exists$ subsequence $\{f_{n_k}\}$ and $f_0 \in L^*$ such that $f_{n_k} \xrightarrow[w*]{} f_0$.

<u>proof:</u> Since $(L, \|\cdot\|)$ is separable, $\exists$ countable dense set $\{x_k\} \subset L$. Know $\exists C < \infty \ni \|f_n\| \leq C$
$\Rightarrow$ $|f_n(x_1)| \leq C$ $\forall n$ $\Rightarrow$ $\{f_n(x_1)\}$ is a bounded sequence in $\mathbb{R}$ $\Rightarrow$ has convergent subsequence. i.e. $\{f_n^{(1)}\} \subseteq \{f_n\}$ and $f_n^{(1)}(x_1) \to f_0(x_1)$.

(Note: this is how we define $f_0(x_1)$!)

Now consider $\{ f_n^{(1)}(x_2) \}$ This is a bounded sequence in $\mathbb{R}$ since $\{ f_n(x_2) \}$ is bounded. $\Rightarrow$ we have a subsequence of $\{f_n^{(1)}\}$ that has

$$f_n^{(2)}(x_2) \longrightarrow f_0(x_2) \qquad \text{(this is how we define } f_0(x_2)\text{!)}$$

Continuing in this way, we find $\{f_n^{(k)}\} \subseteq \{f_n\}$ such that $f_n^{(k)}(x_i) \to f_0(x_i)$ for $i = 1, 2, \dots k$

We now choose our subsequence of $\{f_n\}$ by $f_n^{(n)}$. (diagonalization!) By construction,

$$f_n^{(n)}(x_k) \to f_0(x_k) \quad \forall k.$$

Now, we've constructed f_0 on $\{x_k\}$ ← dense in $(L, \|\cdot\|)$ we want to extend f_0 to all of L, check that $f_0 \in L^*$, and check that $f_n^{(n)}(x) \to f_0(x)$ $\forall x \in L$.

This would prove $f_n^{(n)} \xrightarrow[w*]{} f_0 \in L^*$.

First, I'll abuse notation and rename $\{f_n^{(n)}\}$ by $\{f_n\}$.

$\Rightarrow$ I've got $f_n(x_k) \to f_0(x_k)$ $\forall x_k \in \{x_k\} =: \Delta$.

<u>Step 1</u>: extend f_0 to be linear on span$\{\Delta\}$

<u>Step 2</u>: check that f_0 is a bounded linear fnl on span$\{\Delta\}$

<u>Step 3</u>: extend f_0 to all of L

<u>Step 4</u>: check that f_0 is a bounded linear fnl on L

<u>Step 5</u>: check that $f_n(x) \to f_0(x)$ $\forall x \in L$

Step 1: extend f_0 to be linear on span$\{\Delta\}$.

Let $y \in$ span$\{\Delta\}$. $\Rightarrow$ $y = \sum_1^m \alpha_k x_k$ some $\alpha_k \in \mathbb{R}$, some $x_k \in \{x_n\}$.

$$f_0(y) := \sum_1^m \alpha_k f_0(x_k)$$

This is linear by construction. Furthermore,

$$f_0(y) = \lim_{n\to\infty} f_n(y)$$

since $f_n(y) = \sum_1^m \alpha_k f_n(x_k)$

$$\rightarrow \quad |f_0(y) - f_n(y)| = \left|\sum_1^m \alpha_k (f_0(x_k) - f_n(x_k))\right|$$

$$\leq \sum_1^m |\alpha_k| \, |f_0(x_k) - f_n(x_k)|$$

$$\leq \sum_1^m |\alpha_k| \, \varepsilon \quad \text{for } n \geq N_\varepsilon$$

(and that we had a finite sum here!)

Step 2: check that $|f_0(y)| \leq C\|y\| \quad \forall y \in$ span$\{\Delta\}$

$$|f_0(y)| \leq |f_0(y) - f_n(y)| + |f_n(y)|$$

$$\leq |f_0(y) - f_n(y)| + C\|y\| \quad \text{since } |f_n(y)| \leq \|f_n\| \, \|y\| \leq C\|y\|.$$

fix $\varepsilon > 0$. Choose $N_\varepsilon \ni n \geq N_\varepsilon \Rightarrow |f_0(y) - f_n(y)| < \varepsilon$

$\Rightarrow$ $|f_0(y)| \leq \varepsilon + C\|y\|$ Since $\varepsilon > 0$ was arbitrary,

$|f_0(y)| \leq C\|y\| \quad \forall y \in$ span$\{\Delta\}$.

Step 3: extend f_0 to all of L. Fix $x \in L$

We know $\{x_n\}$ dense in L and L has a norm.

$\Rightarrow$ construct a sequence of the $\{x_n\}$ so that

$$x_{n_k} \to x$$

Since $x_{n_k} \to x$, $\{x_{n_k}\}$ is Cauchy $\Rightarrow \{f_0(x_{n_k})\}$ is Cauchy in $\mathbb{R}$. Why?

$$|f_0(x_{n_k}) - f_0(x_{n_l})| \le C\,\|x_{n_k} - x_{n_l}\|$$ since f_0 bounded l. fnl on span$\{\Delta\}$

$$< C \cdot \varepsilon$$

if $k, l \ge N_\varepsilon$.

$\Rightarrow f_0(x_{n_k}) \to \alpha \in \mathbb{R}$. Call $f_0(x)$ α.

f_0 is linear on L since f_0 is linear on span$\{\Delta\}$.

Step 4: Check that f_0 is bounded linear fnl on L.

using the same x_{n_k} from above,

$$|f_0(x)| \le |f_0(x) - f_0(x_{n_k})| + |f_0(x_{n_k})|$$

$$\le \varepsilon + |f_0(x_{n_k})|$$

if $k \ge N_\varepsilon$

$$\le \varepsilon + C\,\|x_{n_k}\|$$ since f_0 bounded on span$\{\Delta\}$

true $\forall \varepsilon \Rightarrow |f_0(x)| \le C\|x_{n_k}\|$

and $x_{n_k} \to x \Rightarrow \|x_{n_k}\| \to \|x\| \Rightarrow |f_0(x)| \le C\|x\|$

Now that we've shown $f_0 \in L^*$ we just want $f_n \xrightarrow[w*]{} f_0$

Step 5:

$$f_n(x) \to f_0(x) \qquad \forall x \in L.$$

fix $x \in L$.

$$|f_n(x) - f_0(x)| \leq |f_n(x) - f_n(x_k)| + |f_n(x_n) - f_0(x_k)| + |f_0(x_k) - f_0(x)|$$

$$\leq C\|x - x_k\| + |f_n(x_k) - f_0(x_k)| + C\|x - x_k\|$$

since $\{f_n\}, f_0 \in L^*$ w/ norm $\leq C$.

$$\leq 2C\|x - x_k\| + \varepsilon/3$$

if $n \geq N_\varepsilon$ since $f_n(x_k) \to f_0(x_k)$

Note $x_k \in \{x_k\}$ was arbitrary up to now. So I can choose x_k so that $2C\|x - x_k\| < 2\varepsilon/3$ since $\{x_k\}$ dense in L.

$$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \text{if } n \geq N_\varepsilon \text{ and}$$

this proves $f_n(x) \to f_0(x)$ $\forall x \in L$

i.e. $f_n \xrightarrow[w*]{} f_0$.

//

Corr: Assume $(L, \|\cdot\|)$ is separable. Let $M \subseteq L^*$ be strongly bounded. Then M is relatively countably compact in T_{w*}

(i.e. $\exists \{f_n\} \subseteq M$ such that $f_n \xrightarrow[w*]{} f_0 \in [M] \subset L^*$.)

Corr: Let $(L, \|\cdot\|)$ be separable and complete then $M \subseteq L^*$ is strongly bounded if and only if M is relatively countably compact in τ_{w^*}

Note: We will be able to upgrade both of these corrollaries from "countably compact" to "compact". How? By introducing a metric and using the fact that c. compact is the same as compact when there's a metric handy.

Theorem: Let $(L, \| \|)$ be separable. Let

$$S = \{x \in L \mid \|x\| \le 1\}$$

$$S^* = \{f \in L^* \mid \|f\| \le 1\}$$

τ_{w^*} induces a topology on S^* by

$$W \subset S^* \text{ open} \iff W = S^* \cap U \text{ some } U \in \tau_{w^*}$$

Then this induced topology on S^* is the same as the metric topology

$$\rho(f, g) = \sum_1^\infty 2^{-n} | f(x_n) - g(x_n) |$$

where $\{x_n\}_1^\infty \subseteq S$ is dense in S.

Note: We're not metrizing (L^*, τ_{w*}).
We're metrizing $(S^*, S^* \cap \tau_{w*})$.

proof:
1) is ρ a metric?

$$\rho(f,g) = \sum_1^\infty 2^{-n} |f(x_n) - g(x_n)| = \sum_1^\infty 2^{-n} |g(x_n) - f(x_n)| = \rho(g,f).$$

$$\rho(f,g) \le \rho(f,h) + \rho(h,g)$$

follows from $|f(x_n) - g(x_n)| \le |f(x_n) - h(x_n)| + |h(x_n) - g(x_n)|$

Q: if $\rho(f,g)$ finite?

$$\sum_1^m 2^{-n} |f(x_n) - g(x_n)|$$

$$= \sum_1^m 2^{-n} |(f-g)(x_n)|$$

$$\le \sum_1^m 2^{-n} \|f-g\| \, \|x_n\|$$

$$\le \|f-g\| \sum_1^m 2^{-n} \quad \text{since } \|x_n\| \le 1$$

$$\le \|f-g\| \cdot 1$$

$\Rightarrow$ the partial sums are bounded above by $\|f-g\| < \infty$

$$\Rightarrow \sum_1^\infty 2^{-n} |f(x_n) - g(x_n)| < \infty.$$

Non-degeneracy?

$\rho(f,g) = \rho(f-g, 0)$

want $\rho(f,\vec{0}) = 0 \iff f = \vec{0}$. if $f = \vec{0}$ then $\rho(f,\vec{0}) = 0$ is clear.

assume $\rho(f,\vec{0}) = 0 \Rightarrow \sum_1^\infty 2^{-n} |f(x_n)| = 0$

$\Rightarrow f(x_n) = 0$ on a dense subset of S.

since x_n dense in S, this implies $f(x) = 0 \ \forall x \in S$

$\Rightarrow f \equiv 0$ on all of L by $f(\alpha x) = \alpha f(x)$

$\Rightarrow f = \vec{0}$.

$\therefore \rho$ is a metric. We want to prove that

$$(S^*, \rho) = (S^*, S^* \cap \mathcal{T}_{w*})$$

Since $\rho(f-h, g-h) = \rho(f,g)$ rather than looking at the base $\{S(f,\varepsilon)\}$ where $f \in S^*$, it suffices to look at the local base at $\vec{0}$.

1) show that given $\varepsilon > 0$

$$Q_\varepsilon = \{ f \in S^* \mid \rho(f,\vec{0}) < \varepsilon \}$$

$\exists U \in \mathcal{T}_{w*}$ so that $S^* \cap U \subseteq Q_\varepsilon$

this will prove $S^* \cap \mathcal{T}_{w*} \subseteq$ metric topology.

Since Q_ε is a local base for the metric top. at $\vec{0}$.

2) Show that if $\vec{0} \in U \in \tau_{L^*}$ then $\exists\ \varepsilon > 0$ so that

$$Q_\varepsilon = \{ f \in S^* \mid \rho(f, \vec{0}) < \varepsilon \} \subseteq S^* \cap U.$$

this will prove metric topology $\subseteq S^* \cap \tau_{w^*}$

Step 1: Fix $\varepsilon > 0$ Choose N so that $2^{-N} < \varepsilon/2$

Let $U_{x_1 \dots x_N ; \varepsilon/2} = \{ f \in L^* \mid |f(x_1)| < \varepsilon/2 \dots |f(x_N)| < \varepsilon/2 \}$

then $U_{x_1 \dots x_N ; \varepsilon/2} \in \tau_{L^*}$

and if $f \in S^* \cap U_{x_1 \dots x_N ; \varepsilon/2}$ then

$$\rho(f, \vec{0}) = \sum_1^N 2^{-n} |f(x_n)| + \sum_{N+1}^\infty |f(x_n)|\, 2^{-n}$$

$$< \frac{\varepsilon}{2} \sum_1^N 2^{-n} + \sum_{N+1}^\infty 2^{-n} \|f\| \, \|x_n\|$$

$$< \frac{\varepsilon}{2} \sum_1^N 2^{-n} + \sum_{N+1}^\infty 2^{-n} \qquad \text{since } \|f\| \le 1, \ \|x_n\| \le 1$$

$$= \frac{\varepsilon}{4} \frac{1 - (1/2)^N}{1 - 1/2} + 2^{-N-1}$$

$$= \frac{\varepsilon}{2}\left(1 - \left(\frac{1}{2}\right)^N\right) + \frac{1}{2}\left(\frac{1}{2}\right)^N < \frac{\varepsilon}{2} + \frac{1}{2} 2^{-N} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow f \in Q_\varepsilon$ ✓

$\Rightarrow$ found $U \in \tau_{w^*} \ni S^* \cap U \subseteq Q_\varepsilon$.

Step 2: Let $U \in \tau_{w*}$ be a nbd of $\vec{0}$. Then $\exists\, y_1 .. y_m$ and $\delta > 0$ so that

$$U_{y_1 \ldots y_m; \delta} = \{ f \in L^{*} \mid |f(y_1)| < \delta \ldots |f(y_m)| < \delta \}$$

and $U_{y_1 .. y_m; \delta} \subseteq U$.

Now, we can assume $\|y_1\| < 1, \|y_2\| < 1 \ldots \|y_m\| < 1$. WLOG.
Why? if $z_1 = \dfrac{y_1}{2\|y_1\|}$ .. $z_m = \dfrac{y_m}{2\|y_m\|}$ then $\|z_i\| < 1$

and if $\tilde{\delta} = \dfrac{\delta}{2\max\{\|y_1\| .. \|y_m\|\}}$

then $U_{z_1 .. z_m; \tilde{\delta}} \subseteq U_{y_1 .. y_m; \delta}$, so assume $\|y_k\| < 1$ $1 .. m$.

Since $\{X_n\}_1^\infty$ is dense in S, we know $\exists\, X_{n_k}$ so that $\|y_k - X_{n_k}\| < \frac{\delta}{2}$ $k = 1 .. m$

let $N = \max\{n_1 .. n_m\}$ $\quad \varepsilon = \dfrac{\delta}{2^{N+1}}$

if $f \in Q_\varepsilon = \{ f \in S^* \mid \rho(f, \vec{0}) < \varepsilon \}$ then

$$\sum_1^\infty 2^{-n} |f(x_n)| < \varepsilon$$

specifically $2^{-n}|f(x_n)| < \varepsilon$ each $n \Rightarrow |f(x_n)| < \varepsilon 2^n$

$$\Rightarrow |f(x_{n_k})| < \varepsilon \cdot 2^{n_k} \leq \varepsilon 2^N = \frac{\delta}{2}$$

$\Rightarrow$ if $f \in Q_\varepsilon$ then

$$|f(y_k)| \leq |f(x_k)| + |f(x_k - y_k)|$$

$$< \frac{\delta}{2} + \|f\| \, \|x_k - y_k\|$$

$$< \frac{\delta}{2} + 1 \cdot \frac{\delta}{2} \qquad \text{since } \|f\| \leq 1$$

$\Rightarrow f \in U_{y_1 \ldots y_m; \delta}$ and we're done! //

Corr: if $(L \; \|\cdot\|)$ is separable and $M \subset L^*$ is strongly bounded then M is relatively compact.

Proof: M strongly bounded $\Rightarrow \|x\| \leq C < \infty \; \forall x \in M$.

$\Rightarrow \frac{M}{C} \subseteq S^*$. From before $\frac{M}{C}$ is relatively countably compact. i.e. $\left[\frac{M}{C}\right]_{\tau_{w*}}$ is countably compact. but $\left[\frac{M}{C}\right]_{\tau_{w*}} = \left[\frac{M}{C}\right]_\rho$ since $(S^*, S^* \cap \tau_{w*})$ is metrizable and

$\left[\frac{M}{C}\right]_\rho$ is compact. $\Rightarrow \frac{M}{C}$ is relatively compact

$\Rightarrow M$ is relatively compact //

Corr: Let $(L, \|\cdot\|)$ be separable and complete. Then $M \subset L^*$ is strongly bounded if and only if M is relatively compact in τ_{w^*}.