We're now ready to prove our duality theorem for $L^p(\mu)$.

<u>theorem</u>: Let $\frac{1}{p}+\frac{1}{q}=1$. If $1<p<\infty$ then given $\phi\in(L^p)^*$ there exists $g\in L^q$ such that $\phi(f)=\int fg\,d\mu$ for all $f\in L^p$ and hence $(L^p)^*$ is isometrically isomorphic to L^q. The same conclusion holds for $p=1$ if μ is σ-finite.

<u>proof</u>:
First, assume μ is a finite measure. i.e. $\mu(E)<\infty$, where $(E,\mathcal{B},\mu)$ is our measure space. Since μ is finite, the simple functions are in L^p.

Fix $\phi\in(L^p)^*$. Let $\Gamma\in\mathcal{B}$. define $\nu(\Gamma)=\phi(1_\Gamma)$. (Since $1_\Gamma\in L^p$, $\phi(1_\Gamma)$ is well defined. $\phi(1_\Gamma)\in\mathbb{R}$.)

I want to show ν is a signed measure. i.e.
$\nu(\emptyset)=0$, ν assumes at most one value of $+\infty$ and $-\infty$
and if Γ_j is a sequence of disjoint sets in $\mathcal{B}$ then $\nu\left(\bigcup_1^\infty\Gamma_j\right)=\sum_1^\infty\nu(\Gamma_j)$ where the sum converges absolutely if $\nu\left(\bigcup_1^\infty\Gamma_j\right)<\infty$

First $\nu(\emptyset) = 0$. fix $\Gamma \in \mathcal{B}$ then $1_\emptyset + 1_\Gamma = 1_\Gamma$

$$\Rightarrow \underbrace{\phi(1_\emptyset + 1_\Gamma)}_{= \phi(1_\Gamma)} = \phi(1_\emptyset) + \phi(1_\Gamma)$$

$$\Rightarrow \phi(1_\emptyset) = 0 \Rightarrow \nu(\emptyset) = 0 \checkmark$$

Let $\{\Gamma_j\}$ be our disjoint sequence of measurable sets. $\Gamma := \bigcup_1^\infty \Gamma_j \Rightarrow 1_\Gamma = \sum_1^\infty 1_{\Gamma_j}$

where $\sum_1^n 1_{\Gamma_j} \to 1_\Gamma$ as $n \to \infty$ in the L^p norm.

because $\| 1_\Gamma - \sum_1^n 1_{\Gamma_j} \|_{L^p} = \| \sum_{n+1}^\infty 1_{\Gamma_j} \|_{L^p}$

$$= \sqrt[p]{\mu\left(\bigcup_{n+1}^\infty \Gamma_j\right)} \quad (\text{since } p < \infty)$$

and $\mu\left(\bigcup_{n+1}^\infty \Gamma_j\right) \to 0$ as $n \to \infty$ since $\mu(E) < \infty$

$\therefore$ since $\phi \in (L^p)^*$,

$$\sum_1^n \nu(\Gamma_j) = \phi\left(\sum_1^n 1_{\Gamma_j}\right) \to \phi(1_\Gamma) = \nu\left(\bigcup_1 \Gamma_j\right)$$

Thus $\sum_{1}^{n} \nu(\Gamma_j) \to \nu\left(\bigcup_{1}^{\infty} \Gamma_j\right)$ as $n \to \infty$.

(Check the absolute convergence on your own!)

Thus ν is a signed measure. Now I want to show $\nu \ll \mu$. Let $\Gamma \in \mathcal{B}$ so that $\mu(\Gamma) = 0$. Then $1_\Gamma = 0$ almost everywhere $\Rightarrow$ $1_\Gamma =$ the zero function in L^p.

$\Rightarrow \phi(1_\Gamma) = 0 \Rightarrow \nu(\Gamma) = 0$. This shows $\nu \ll \mu$.

By the Radon Nikodym theorem $d\nu = g\,d\mu$ for some $g \in L^1(\mu)$. i.e. $\nu(\Gamma) = \int_\Gamma g\,d\mu$ for any $\Gamma \in \mathcal{B}$.

$\Rightarrow \phi(1_\Gamma) = \int_E g 1_\Gamma \, d\mu$ for any $\Gamma \in \mathcal{B}$.

$\Rightarrow \phi(f) = \int_E fg\,d\mu$ for any simple function f.

and $\left|\int_E fg\,d\mu\right| = |\phi(f)| \le \|\phi\|_{(L^p)^*} \|f\|_{L^p}$

$\Rightarrow M_q(g) = \sup\left\{ \left|\int_E fg\,d\mu\right| \;\middle|\; f \text{ simple}, \|f\|_{L^p} = 1 \right\} < \infty$.

by earlier theorem, this implies $g \in L^q$ and

$M_q(g) = \|g\|_{L^q}$.

Now that we know $g \in L^q$, we know

that $\psi(f) := \int fg \, d\mu$ where $f \in L^p$

is definitely in $(L^p)^*$ and $\psi = \phi$ on a dense set (the simple functions). Since ϕ and ψ are both continuous this means $\phi = \psi$.

i.e. $\phi(f) = \int fg \, d\mu$ for all $f \in L^p$. ✓

Now assume μ is σ-finite. Let $\{E_n\}$ be an increasing sequence of sets such that $0 < \mu(E_n) < \infty$ and $E = \bigcup_1^\infty E_n$.

Given E_n, $L^p(E_n) := \{f \in L^p(E) \mid f = 0 \text{ outside } E_n\}$

Similarly define $L^q(E_n)$.

By above for each n, $\exists\, g_n \in L^q(E_n)$ such that $\phi(f) = \int f g_n \, d\mu \quad \forall f \in L^p(E_n)$ and

$$\|g_n\|_{L^q} = \|\phi|_{L^p(E_n)}\| \leq \|\phi\|$$

g_n is determined uniquely up to sets of measure zero $\Rightarrow$ if $m < n$ then $g_n = g_m$ a.e. in E_m

Therefore we define g on E by

$$g(x) := g_{n_0}(x) \text{ where } n_0 = \{\inf m \mid x \in E_m\}$$

And g_n increases to g almost everywhere in E.

$\Rightarrow$ by monotone convergence theorem,

$$\|g\|_{L^q} = \lim_{n\to\infty} \|g_n\|_{L^q} \le \|\phi\|$$

$\Rightarrow g \in L^q(E)$.

If $f \in L^p$ then $f 1_{E_n} \to f$ pointwise and therefore $f 1_{E_n} \to f$ in L^p by Lebesgue Dominated convergence theorem. Since ϕ is continuous

$$\phi(f 1_{E_n}) \to \phi(f) \quad \text{as } n \to \infty$$

$\Rightarrow$ thus $\phi(f) = \lim_{n\to\infty} \phi(f 1_{E_n}) = \lim_{n\to\infty} \int_{E_n} fg\, d\mu = \int fg\, d\mu$.

So we've found our $g \in L^q$ so that $\phi(f) = \int fg\, d\mu \ \forall f \in L^p$.

We're done with the case of

$1 \le p < \infty$ and μ σ-finite.

This leaves the case of $1 < p < \infty$ and μ (not assumed σ-finite).

Assume $1 < p < \infty$. By above, if $\Gamma \subset \mathcal{B}$ and Γ is σ-finite then $\exists$ almost everywhere unique $g_\Gamma \in L^q(\Gamma)$ so that $\phi(f) = \int f g_\Gamma \, d\mu$ for all $f \in L^p(\Gamma)$ and $\|g_\Gamma\|_{L^q} \leq \|\phi\|_{(L^p)^*}$.

$$M := \sup \{ \|g_\Gamma\|_{L^q} \mid \Gamma \in \mathcal{B}, \Gamma \ \sigma\text{-finite} \}$$

then $M \leq \|\phi\|_{(L^p)^*}$. Let $\{\Gamma_n\} \subset \mathcal{B}$ be a sequence such that $\|g_{\Gamma_n}\|_{L^q} \to M$ and define

$$\Gamma = \bigcup_1^\infty \Gamma_n.$$

then Γ is also σ-finite and $\|g_{\Gamma_n}\|_{L^q} \leq \|g_\Gamma\|_{L^q} = M$

Let A be any σ-finite set that contains Γ. Then

$$g_A = g_\Gamma + g_{A-\Gamma}$$

$$\Rightarrow \int |g_\Gamma|^q d\mu + \int |g_{A-\Gamma}|^q = \int |g_A|^q \leq M^q = \int |g_\Gamma|^q$$

recall g_Γ and $g_{A-\Gamma}$ have disjoint support

at this point, we used $q < \infty$

$$\Rightarrow g_{A-\Gamma} = 0 \text{ almost everywhere in } A - \Gamma$$

$\Rightarrow g_A = g_\Gamma$ almost everywhere.

But if $f \in L^p$ then $A = \Gamma \cup \{f \neq 0\}$ is σ-finite

$$\Rightarrow \phi(f1_{\{f\neq 0\}}) = \int f1_{\{f\neq 0\}} g_A = \int f1_{\{f\neq 0\}} g_\Gamma \, d\mu$$

$$= \int f g_\Gamma \, d\mu$$

and $\phi(f) = \phi(f1_{\{f\neq 0\}} + f1_{\{f=0\}})$

$= \phi(f1_{\{f\neq 0\}})$ since $f1_{\{f=0\}}$ is the zero function

$\Rightarrow \phi(f) = \int f g_\Gamma \, d\mu \Rightarrow$ we can take $g = g_\Gamma$ and we're done! //

Corr: if $1<p<\infty$ then L^p is reflexive

To finish up, let's look at the $p=1$ and $p=\infty$ cases.

First $p=1$. What do we know about $(L^1)^*$? We know that $g \in L^\infty \Rightarrow \Psi_g \in (L^1)^*$ but we needed μ semifinite in order to get that this mapping was an isometry. Isometry $\Rightarrow$ the mapping is 1:1.

So can we find an ex. where μ isn't semifinite and the mapping isn't 1:1?

<u>theorem</u>: if μ is not semifinite then $L^\infty \hookrightarrow (L^1)^*$ isn't injective.

<u>proof</u>:

Assume $\Gamma \subset E$ is a set of infinite measure that contains no subset $A \in \mathcal{B}$ so that $\mu(A) \in (0,\infty)$.

Let $f \in L^1$. Then $\{f \neq 0\}$ is a σ-finite set and therefore $\{f \neq 0\} \cap \Gamma$ is a set of measure zero (If it weren't a set of measure zero then we would use $\{f \neq 0\} = \bigcup_1^\infty \Gamma_n$ where $\mu(\Gamma_n) \in (0,\infty)$ to find a subset of Γ with measure $\in (0,\infty)$.)

Now we demonstrate our failure of injectivity.

Let $1_\Gamma \in L^\infty$. Then $\Psi_{1_\Gamma} \in (L^1)^*$.

But $\Psi_{1_\Gamma}(f) = \int f 1_\Gamma \, d\mu = \int_\Gamma f \, d\mu = 0$ for all $f \in L^1$ (by above)

$\Rightarrow \Psi_{1_\Gamma} \equiv 0$ even though $1_\Gamma \neq 0$ in L^∞. This shows the embedding of L^∞ into $(L^1)^*$ is not 1:1

example where $L^\infty \hookrightarrow (L^1)^*$ is not onto:

Let E be an uncountable set.

take $\mathcal{B}$ = the power set of E and μ the counting measure. $\Rightarrow$ $(E, \mathcal{B}, \mu)$ is a measure space.

Now let $\mathcal{B}_0 \subset \mathcal{B}$ be the collection of countable subsets of E and the subsets whose complement is countable. i.e.

$$\mathcal{B} \supseteq \mathcal{B}_0 = \{A \subset E \mid A \text{ or } E \setminus A \text{ is countable}\}$$

then $\mathcal{B}_0$ is a σ-algebra and $\mu_0 = \mu|_{\mathcal{B}_0}$ is a measure

If $f \in L^1(\mu)$ then f vanishes outside a countable set in E.
$\Rightarrow$ $L^1(\mu) \subseteq L^1(\mu_0)$. And $L^1(\mu_0) \subseteq L^1(\mu)$ since μ_0 is a restriction of μ. $\Rightarrow$ $L^1(\mu) = L^1(\mu_0)$.

On the other hand, $L^\infty(\mu)$ = bounded functions on E while $L^\infty(\mu_0)$ = bounded functions that are constant except on a countable set.

$\Rightarrow$ $L^\infty(\mu_0) \subseteq L^\infty(\mu)$. An exercise shows that $(L^1(\mu_0))^* = L^\infty(\mu)$
$\Rightarrow$ $L^\infty(\mu_0) \hookrightarrow (L^1(\mu_0))^*$ is not onto

For $p=\infty$ we know that $L^1 \hookrightarrow (L^\infty)^*$ is always 1:1 (we proved this) but it is rarely onto.

<u>Example where $L^1 \hookrightarrow (L^\infty)^*$ is not onto:</u>

Let $E=[0,1]$ and $\mu=$ Lebesgue measure restricted to E.

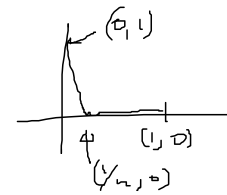
$$\psi_0(f) = f(0)$$

is a bounded linear functional on $C([0,1])$ and $C([0,1])$ is a subspace of L^∞. By Hahn-Banach theorem, we can extend ψ_0 to all of $L^\infty([0,1])$.

I claim $\nexists\, g \in L^1([0,1])$ so that

$$\psi(f) = \int fg \, dx \quad \text{for all } f \in L^\infty.$$

Take $f_n \in C([0,1])$ $\quad f_n := \max\{0, 1-nx\}$.



← f_n's graph

then $\psi(f_n) = 1 \quad \forall n.$

but $f_n(x) \to 0$ for all $x \in (0,1] \Rightarrow$ by dominated convergence theorem $\int f_n g \, dx \to 0$ if $g \in L^1$.

$\Rightarrow \nexists\, g \in L^1$ so that $\psi = \psi_g$.