What is the dual of L^p?

By Hölder's inequality, if $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\int |fg| \leq \|f\|_p \|g\|_q$$

$\therefore$ Since $\left|\int fg \, d\mu\right| \leq \int |fg| \, d\mu$

we see that $\Psi_g(f) := \int fg \, d\mu$ is a bounded linear functional on L^p if $g \in L^q$. And

$$\|\Psi_g\|_{(L^p)^*} \leq \|g\|_q$$

In this way, we can map $L^q(\mu)$ into $(L^p(\mu))^*$.

i.e. $g \to \Psi_g$.

When we did this for Hilbert spaces we found this map was an isometry: $\|\Psi_g\|_{(L^p)^*} = \|g\|_{L^q}$.

Is this still the case?

<u>Note</u>: For L^2 we usually define $\Psi_g(f) = \int f \bar{g} \, d\mu$ if f and g are complex-valued. If they're real-valued, it doesn't make a difference. Either way, whether $\Psi_g := \int fg \, d\mu$ or $\Psi_g(f) = \int f\bar{g} \, d\mu$, the following results hold.

theorem: Suppose $\frac{1}{p}+\frac{1}{q}=1$ and $1 \le q < \infty$

if $g \in L^q$ then

$$\|g\|_{L^q} = \|\Psi_g\|_{(L^p)^*} = \sup \left\{ \left| \int f g \, d\mu \right| \;\middle|\; \|f\|_{L^p} = 1 \right\}$$

If μ is semifinite then the result also holds for $q = \infty$.

So this proves that $L^q \hookrightarrow (L^p)^*$ is an isometry for $p \in (1, \infty]$ and for $p = 1$ if μ is semifinite.

defn: $(E, \mathcal{B}, \mu)$. μ is semifinite if for each $\Gamma \in \mathcal{B}$ with $\mu(\Gamma) = \infty$, $\exists$ $F \in \mathcal{B}$ with $F \subseteq \Gamma$ so that $0 < \mu(F) < \infty$.

(need to exclude $F = \emptyset$. ☺)

Proof: By Hölder, we know $\|\Psi_g\|_{(L^p)^*} \le \|g\|_{L^q}$

If $g = 0$ almost everywhere then $\|\Psi_g\| = \|g\| = 0$.

So assume $g \ne 0$.

if $1 < q < \infty$ define

$$f = \frac{|g|^{q-1}\,\overline{\mathrm{sgn}(g)}}{\|g\|_{L^q}^{q-1}}$$

$$\Rightarrow \left(\|f\|_{L^p}\right)^p = \frac{\int |g|^{(q-1)p}\,d\mu}{\left(\|g\|_{L^q}\right)^{(q-1)p}}$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \frac{1}{p} = \frac{q-1}{q} \Rightarrow q = p(q-1)$$

$$= \frac{\int |g|^q\,d\mu}{\left(\|g\|_{L^q}\right)^q} = 1 \qquad \Rightarrow \|f\|_{L^p} = 1 \quad \left(\begin{matrix}\text{valid}\\ \text{test fxn}\end{matrix}\right)$$

$$\Rightarrow \|\Psi_g\| \geq \left|\int f g\,d\mu\right| = \frac{\int |g|^q\,d\mu}{\left(\|g\|_{L^q}\right)^{q-1}} = \|g\|_{L^q}$$

$$\Rightarrow \|\Psi_g\|_{(L^p)^*} = \|g\|_{L^q} \ \checkmark$$

if $q = 1$ take $f = \overline{\mathrm{sgn}(g)}$ $\Rightarrow \|f\|_{L^p} = \|f\|_{L^\infty} = 1$ $\left(\begin{matrix}\text{valid}\\ \text{test fxn}\end{matrix}\right)$

and $\|\Psi_g\| \geq \left|\int f g\,d\mu\right| = \left|\int |g|\,d\mu\right| = \|g\|_{L^1}$

$$\Rightarrow \|\Psi_g\|_{(L^\infty)^*} = \|g\|_{L^1}$$

if $q = \infty$ for $\varepsilon > 0$ define $A_\varepsilon = \{ x \mid |g(x)| > \|g\|_\infty - \varepsilon \}$
then $\mu(A_\varepsilon) > 0$ by defn of ess-sup $\|g\|_\infty$
$\Rightarrow$ since μ is semifinite $\exists$ $B \in \mathcal{B}$ with $B \subseteq A_\varepsilon$ and
$0 < \mu(B) < \infty$ (if $\mu(A_\varepsilon) < \infty$ just take $B = A_\varepsilon$)

Here's our test function:

$$f := \frac{1}{\mu(B)} \overline{\operatorname{sgn}(g)} \, 1_B$$

$$\Rightarrow \|f\|_{L^1} = \frac{1}{\mu(B)} \int |\operatorname{sgn} \bar{g}| 1_B = \frac{\mu(B)}{\mu(B)} = 1$$ (valid test fxn.)

$$\Rightarrow \|\Psi_g\|_{(L^1)^*} \geq \left| \int fg \right| = \frac{1}{\mu(B)} \int_B |g| \, d\mu \geq \frac{1}{\mu(B)} \int_B \|g\|_\infty - \varepsilon \, d\mu$$ since $B \subseteq A_\varepsilon$

$$= \|g\|_\infty - \varepsilon.$$

$\Rightarrow$ $\|\Psi_g\|_{(L^1)^*} \geq \|g\|_\infty - \varepsilon$. Since ε was arb, we take $\varepsilon \downarrow 0$

and $\|\Psi_g\|_{(L^1)^*} \geq \|g\|_\infty$ $\Rightarrow$ $\|\Psi_g\|_{(L^1)^*} = \|g\|_{L^\infty}$. //

So we've shown that

$$g \in L^q \Rightarrow \quad f \to \int fg\,d\mu \quad \text{is in } (L^p)^*$$

is it true that

$$f \to \int fg\,d\mu \in (L^p)^* \Rightarrow g \in L^q \quad ?$$

<u>theorem</u>: Assume $\frac{1}{p}+\frac{1}{q}=1$. Suppose that g is a measurable function on X such that $fg \in L^1(\mu)$ for all $f \in \{$simple functions that vanish outside a set of finite μ.$\} = S$ and the quantity

$$M_q(g) = \sup\left\{ \left|\int fg\right| \;\middle|\; f \in S \text{ and } \|f\|_{L^p} = 1 \right\}$$

is finite. Further suppose either that $S_g = \{g \neq 0\}$ is σ-finite or that μ is semifinite. Then $g \in L^q$ and $M_q(g) = \|g\|_{L^q}$.

<u>Note</u>: our simple functions have finite values, by assump.

<u>Defn</u>: $\Gamma \in \mathcal{B}$ is σ-finite if $\Gamma = \bigcup_{i}^{\infty} \Gamma_i$ where $\Gamma_i \in \mathcal{B}$ $\forall i$ and $\mu(\Gamma_i) < \infty$ $\forall i$

Proof:

First note that if f is a bounded measurable function that vanishes outside a set of finite measure and $\|f\|_{L^p}=1$ then $\left|\int fg\,d\mu\right| \le M_q(g)$.

Why? Take our (usual) sequence of simple functions ϕ_n so that $|\phi_n| \le |f|$ and $\phi_n \to f$ almost everywhere.
ϕ_n will vanish outside the same set that f vanishes outside of $\Rightarrow \phi_n \in S$.

$$|\phi_n| \le \|f\|_{L^\infty} 1_E \qquad E = \text{set that } f \text{ vanishes outside of}$$

and $1_E \in S \Rightarrow 1_E g \in L^1$

$\Rightarrow \quad |\phi_n g| \le \|f\|_{L^\infty} |g| 1_E \leftarrow L^1$ function. We can therefore apply Lebesgue dominated convergence theorem $\Rightarrow \displaystyle\lim_{n\to\infty} \int \phi_n g\,d\mu = \int \lim_{n\to\infty} \phi_n g\,d\mu = \int fg\,d\mu$

and so $\displaystyle\left|\int fg\,d\mu\right| = \lim_{n\to\infty} \left|\int \phi_n g\,d\mu\right| \le M_q(g)$ since $\phi_n \in S$.

Now assume $q < \infty$. We can assume $S_g = \{g \neq 0\}$ is σ-finite. (Either this is true by hypothesis, or μ is semifinite in which case we can prove it's true. See lemma.)

Let $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$ such that $S_g = \bigcup_1^\infty E_n$ and $\mu(E_n) < \infty$ for each n. Let ϕ_n be a sequence of simple functions such that $\phi_n \to g$ pointwise and $|\phi_n| \leq |g|$. (Our usual family of ϕ_n.)

Define $g_n := 1_{E_n} \phi_n$. Then $g_n \to g$ pointwise

and $|g_n| \leq g$ and g_n vanishes outside E_n (a set of finite meas.)

$$f_n := \frac{|g_n|^{q-1} \overline{\operatorname{sgn}(g)}}{\|g_n\|_q^{q-1}} \qquad \text{(our test functions)}$$

as before, $\|f_n\|_{L^p} = 1$. Also, since f_n is a bounded measurable function that vanishes outside a set of measure finite, from before $|\int f_n g \, d\mu| \leq M_q(g)$.

We apply Fatou's Lemma to find

$$\|g\|_{L^q} \leq \varliminf_{n\to\infty} \|g_n\|_{L^q} = \varliminf_{n\to\infty} \int |f_n g_n| \, d\mu \qquad \text{by our clever choice of } f_n$$

$$\leq \varliminf_{n\to\infty} \int |f_n g| \, d\mu = \varliminf_{n\to\infty} \int f_n g \, d\mu \leq M_q(g)$$

(since $g_n \to g$) (since $f_n g \geq 0$)

we already know $M_q(g) \leq \|g\|_{L^q}$ by Hölder so now we've got $g \in L^q$ and $M_q(g) = \|g\|_{L^q}$.

So we've finished the proof in the $q < \infty$ case.

Now assume $q = \infty$. We're told

$$M_\infty(g) := \sup\left\{ |\textstyle\int fg| \;\middle|\; f \in S \text{ and } \|f\|_{L^1} = 1 \right\} < \infty.$$

Fix $\varepsilon > 0$

Let $A_\varepsilon := \{ x \mid |g(x)| \geq M_\infty(g) + \varepsilon \}$.

If we can show $\mu(A_\varepsilon) = 0$ then this will imply (by the defn of ess-sup) that $\|g\|_{L^\infty} \leq M_\infty(g)$.

Assume $\mu(A_\varepsilon) > 0$. Then $\exists$ $B \subseteq A_\varepsilon$ with $0 < \mu(B) < \infty$ (since $A \subseteq S_g$ which is σ-finite by hypothesis or since μ is semifinite by hypothesis.)

$$f := \frac{1}{\mu(B)} \mathbb{1}_B \overline{\operatorname{sgn}(g)} \qquad \text{(our valid test fxn)}$$

has $\|f\|_{L^1}$ and $\int fg = \frac{1}{\mu(B)} \int_B |g| > M_\infty(g) + \varepsilon$

since $B \subseteq A_\varepsilon$. But f is a bounded measurable function that vanishes outside a set of finite measure so by the first part, $|\int fg| \leq M_\infty(g)$. $\times$

This proves $\mu(A_\varepsilon) = 0 \Rightarrow M_\infty(g) \leq \|g\|_{L^\infty}$.

$\|g\|_{L^\infty} \geq M_\infty(g)$ is immediate:

$$M_\infty(g) = \sup \left\{ \left| \int fg \right| \;\middle|\; f \in S \text{ and } \|f\|_{L^1} = 1 \right\}$$

and $\left| \int fg \right| \leq \int |fg| \leq \|g\|_{L^\infty} \int |f| = 1$

This proves $\|g\|_{L^\infty} = M_\infty(g) < \infty \Rightarrow$

$g \in L^\infty(\mu)$ as desired //

<u>lemma</u>: if μ is semifinite, $q < \infty$ and $M_q(g) < \infty$ then $\{|g| > \varepsilon\}$ has finite measure for all $\varepsilon > 0$ and hence $S_g = \{g \neq 0\}$ is σ-finite

<u>proof</u>:

Exercise!

<u>theorem</u>: Assume $\frac{1}{p}+\frac{1}{q}=1$. If $1<p<\infty$ then for each $\phi \in (L^p)^*$ there exists $g \in L^q$ such that $\phi(f) = \int fg \, d\mu$ for all $f \in L^p$. Hence L^q is isometrically isomorphic to $(L^p)^*$. The same result holds for $p=1$ provided μ is σ-finite

<u>Defn</u>: given $(E, \mathcal{B}, \mu)$, μ is σ-finite if $E = \bigcup_1^\infty E_j$ where $\mu(E_j)<\infty$ $j=1..\infty$.

<u>Note</u>: $\mathbb{R}^n$ w/ Lebesgue measure is σ-finite. So the theorem tells us $(L^p)^* \simeq L^q$ for $1 \le p < \infty$.

<u>Corr</u>: L^p is reflexive if $1<p<\infty$.

We will need to make a detour to complex measures and the Radon-Nikodym theorem.
Until now, all our measures have been

$$\mu : \mathcal{B} \to [0, \infty]$$

i.e. "positive" measures. if $\mu : \mathcal{B} \to [0, \infty)$ then they're "finite" measures.

A useful concept will be the signed measures.

$$\nu : \mathcal{B} \to \overline{\mathbb{R}}$$

such that 1) $\nu(\phi) = 0$

2) ν assumes at most one of $\pm\infty$

3) if $\{E_k\}_1^\infty$ is a seq of disjoint sets in $\mathcal{B}$ then $\nu\left(\bigcup_1^\infty E_k\right) = \sum_1^\infty \nu(E_k)$ and $\sum_1^\infty |\nu(E_k)| < \infty$. if $\nu\left(\bigcup_1^\infty E_k\right) < \infty$

Note: if μ is a (positive) measure and $f \in L^1(\mu)$, then $f\,d\mu$ is a signed measure.

<u>ex 2</u>: if μ_1 and μ_2 are (positive) measures and at least one of them is finite then $\nu = \mu_1 - \mu_2$ is a signed measure.

<u>ex 3</u>: If μ is a measure and $f: E \to \overline{\mathbb{R}}$ is a measurable function such that

$$\int f_+ \, d\mu < \infty \quad \text{or} \quad \int f_- \, d\mu < \infty$$

then $\nu(\Gamma) := \int_\Gamma f \, d\mu$ is a signed measure.

given two signed measures one can define

<u>defn</u>: ν is singular <u>with respect</u> to μ ($\nu \perp \mu$) if $\exists\, E_1, E_2 \in \mathcal{B}$ so that $E_1 \cap E_2 = \emptyset$ $E_1 \cup E_2 = E$ such that $\nu(\Gamma) = 0 \ \forall \Gamma \subseteq E_1$ and $\mu(\Gamma) = 0 \ \forall \Gamma \subseteq E_2$.

i.e. ν and μ live on disjoint sets.

One can prove

Jordan Decomposition theorem: If ν is a signed measure, there exist unique positive measures ν^+ and ν^- such that

$$\nu = \nu^+ - \nu^- \quad \text{and} \quad \nu^+ \perp \nu^-$$

defn: Suppose ν is a signed measure on $(E, \mathcal{B})$ and μ is a (positive) measure on $(E, \mathcal{B})$. Then ν is absolutely continuous wrt μ $(\nu \ll \mu)$ if $\mu(\Gamma) = 0 \Rightarrow \nu(\Gamma) = 0$.

Note: $\nu \perp \mu$ and $\nu \ll \mu \Rightarrow \nu = 0$

The Lebesgue-Radon-Nikodym theorem: Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(E, \mathcal{B})$. Then exist unique σ-finite signed measures λ, ρ on $(E, \mathcal{B})$ such that

$$\lambda \perp \mu \;, \quad \rho \ll \mu \quad \underline{\text{and}} \quad \nu = \lambda + \rho$$

Moreover, $\exists$ $f: E \to \mathbb{R}$ with $\int f^+ d\mu < \infty$ or $\int f^- d\mu < \infty$ such that $d\rho = f\, d\mu$. And any two such functions are equal almost everywhere.

In the special case where $\nu \ll \mu$, the Lebesgue-Radon-Nikodym theorem says

$$d\nu = f\,d\mu$$

for some f. f is called the Radon Nikodym derivative of ν with respect to μ. And the existence of f is called the Radon Nikodym theorem.

We need the Radon Nikodym theorem to prove

$$(L^p)^* \hookleftarrow L^q \quad \text{for} \quad 1 < p < \infty$$

(for $p=1$ if μ is σ-finite.)