$$\int_1^\infty f'(x) \sin(2k\pi x)dx = O(1/k)$$

which completes the proof.

Thus for $1<a<2$ the series $\sum_{x=1}^\infty f(x)$ converges if $b-a+1>0$ ($a+b>1$ is satisfied). Note that convergence in case $a=1$ does not depend on any conditions, since $a+b>1$ and $b-a+1>0$ are always satisfied.

The example shows that the scope of applications of Corollary 2 is considerably wider than that of Corollary 1.

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MATHEMATICAL NOTES

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THERE IS AN ELEMENTARY PROOF OF PEANO'S EXISTENCE THEOREM

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A "Research Problem" by H. C. Kennedy (see [2]) stimulated the author to publish the following proof, which he discovered more than 10 years ago and since has used in lectures. We shall first present the proof, which is constructive and deserves, in our opinion, the adjective "elementary" without restriction, because the construction of a monotone decreasing sequence of approximate solutions avoids equicontinuous families and an appeal to the Ascoli lemma. I am hesitant in making the same statement on Peano's or Perron's proof. Critical and historical remarks will be given at the end of the paper.

PEANO'S EXISTENCE THEOREM. *Let J be the interval $0 \leqq t \leqq T (T>0)$ and let $f(t, x): J\times \mathbf{R} \to \mathbf{R}$ be a continuous and bounded function. Then there exists, for a given $u_0 \in \mathbf{R}$, at least one continuously differentiable function $u(t): J \to \mathbf{R}$ satisfying*

$$u' = f(t, u) \quad \text{in} \quad J \quad \text{and} \quad u(0) = u_0. \tag{1}$$

The proof utilizes a variant of the Euler-Cauchy polygon method. Let $h = T/n > 0$ be given and $t_i = ih (i = 0, 1, \cdots, n)$. In the polygon method one constructs a sequence $(v_i)_0^n$ according to

$$v_0 = u_0, \qquad v_{i+1} = v_i + hf(t_i, v_i) \quad (i = 0, 1, \cdots, n-1).$$

We use instead the formula

(2) $v_0 = u_0, \quad v_{i+1} = v_i + h \max\{f(t, x): t_i \leqq t \leqq t_{i+1}, v_i - 3Mh \leqq x \leqq v_i + Mh\},$

where $|f(t, x)| \leqq M$ in $J \times R$; as in the classical Euler-Cauchy polygon method the approximate solution $v(t)$ is constructed by joining the points (t_i, v_i) $(i = 0, 1, \cdots, n)$ by a polygonal line. This construction, carried out for the parameters $h, h/2, h/4, \cdots$, leads to a monotone decreasing sequence of approximate solutions.

In proving the last statement we use the following notation: $h = T/n > 0$ is fixed, $t_i = ih$ $(i = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, n)$, $(v_i) = (v_0, v_1, \cdots, v_n)$ is constructed according to (2) with respect to the parameter h, $(w_i) = (w_0, w_{1/2}, w_1, w_{3/2}, \cdots, w_n)$ is constructed similarly, but with respect to the parameter $h/2$, $v(t)$ and $w(t)$ are continuous, piecewise linear functions satisfying $v(t_i) = v_i$ for $i = 0, 1, 2, \cdots, n$ and $w(t_i) = w_i$ for $i = 0, \frac{1}{2}, 1, \cdots, n$. Let $R(t, x; h)$ be the rectangle $[t, t+h] \times [x - 3Mh, x + Mh]$ and let $R_i = R(t_i, v_i; h)$ $(i = 0, 1, \cdots, n)$, $S_i = R(t_i, w_i; h/2)$ $(i = 0, \frac{1}{2}, 1, \cdots, n)$. In this notation the sequences (v_i) and (w_i) are defined as follows:

$$v_0 = w_0 = u_0, \qquad v_{i+1} = v_i + h \max f(R_i), \qquad w_{i+1/2} = w_i + (h/2) \max f(S_i).$$

Here the standard notation $\max f(A) = \max\{f(t, x): (t, x) \in A\}$ was used.

We shall prove, by induction on i, that $w(t) \leqq v(t)$ in J. Let us assume that i is a nonnegative integer and that $w(t) \leqq v(t)$ for $0 \leqq t \leqq t_i$. Then there are two cases to be considered,

$$\text{(a) } w_i \leqq v_i - Mh; \qquad \text{(b) } v_i - Mh < w_i \leqq v_i.$$

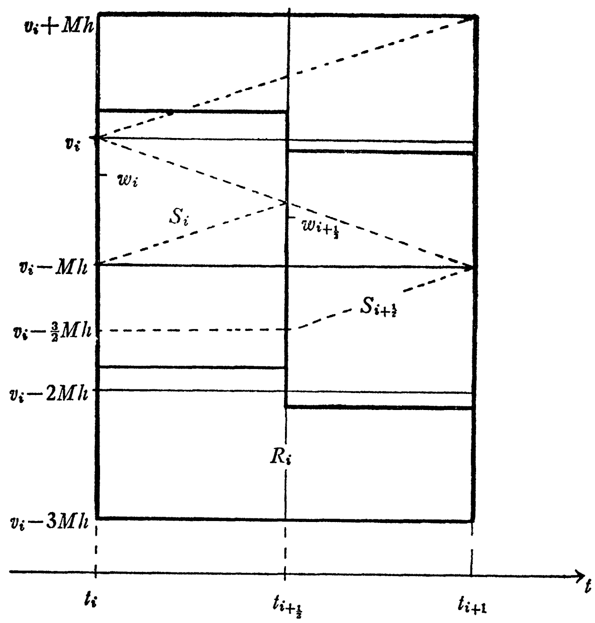
In case (a) we have $w(t) \leqq v(t)$ for $t_i \leqq t \leqq t_i + h/2$, since $|v'|, |w'| \leqq M$. In case (b) we have $S_i \subset R_i$ and hence $w' \leqq v'$ for $t_i < t < t_i + h/2$; again the inequality $w \leqq v$ in $[t_i, t_{i+1/2}]$ follows. In essentially the same way one shows that $w \leqq v$ in $[t_{i+1/2}, t_{i+1}]$. Since $v(t_{i+1/2}) \leqq v_i + Mh/2$ there are two cases

$$\text{(a) } w_{i+1/2} \leqq v_i - \tfrac{3}{2}Mh; \qquad \text{(b) } v_i - \tfrac{3}{2}MH < w_{i+1/2} \leqq v_i + \tfrac{1}{2}Mh.$$

In case (a) it follows as above that $w(t) \leqq v(t)$ for $t_{i+1/2} \leqq t \leqq t_{i+1}$, while in case (b) we have $S_{i+1/2} \subset R_i$ and therefore $w'(t) \leqq v'(t)$ for $t_{i+1/2} < t < t_{i+1}$. In the figure case (a) is indicated by the punctuated lines, while the two rectangles S_i, $S_{i+1/2}$ correspond to case (b). So far we have proved that the inequality $w(t) \leqq v(t)$ holds in $[0, t_{i+1}]$; it follows then by induction that this inequality is true in J.

The rest of the proof is a matter of routine. Let $h_k = T \cdot 2^{-k}$, $k = 1, 2, \cdots$, and let $v_k(t)$ be the piecewise linear, continuous function constructed according to (2) with respect to $h = h_k$. We have proved that $v_{k+1}(t) \leqq v_k(t)$ in J. Furthermore, v_k satisfies a Lipschitz condition with a Lipschitz constant M independent of k, and $v_k(t)$ is bounded below by $u_0 - Mt$. Therefore $u(t) = \lim_{k \to \infty} v_k(t)$ exists, the convergence being uniform in J. The function u is continuous (even Lipschitz continuous) in J; it is the desired solution of the initial value problem, as it will be shown now.

With the exception of a finite number of points, $v_k'(t)$ exists and, according to (2),



$$v_k'(t) = f(t', x'), \quad \text{where} \quad |t - t'| \leqq h_k, \qquad |v_k(t) - x'| \leqq 3Mh_k.$$

If $d(s)$ denotes a modulus of continuity for f,

$$|f(t, x) - f(t', x')| \leqq d(|t - t'| + |x - x'|) \quad \text{in } J \times [u_0 - MT, u_0 + MT],$$

then $v_k'(t) = f(t, v_k(t)) + \alpha_k(t)$, where $|\alpha_k(t)| \leqq d(h_k + 3Mh_k)$ and hence

$$(3) \qquad v_k(t) = u_0 + \int_0^t f(s, v_k(s))ds + \beta_k(s), \quad |\beta_k(s)| \leqq d(h_k + 3Mh_k)T.$$

It follows immediately from (3) for $k \to \infty$ that

$$u(t) = u_0 + \int_0^t f(s, u(s))ds \qquad \text{in } J,$$

i.e., that u is a solution of the initial value problem (1).

REMARKS. (a) It is an easy exercise to prove the following statement: If $\bar{u}(t)$ is another solution of (1), then $\bar{u} \leqq v_k$ in J. Hence the solution u constructed above is indeed the maximal solution.

(b) It is not the aim of this paper to investigate the existence proofs by Peano (1886 and 1890) and Perron (1915). Nevertheless, the author is not in agreement with several critical remarks in [2] concerning these proofs. Perron's proof is correct. Furthermore the remark in [2] that Peano's second proof (1890) is based on successive approximation is incorrect.

(c) The theoretical basis for the proof given in this paper as well as for

Peano's first proof (1886) and Perron's proof is a theorem on differential inequalities which in the simplest case reads as follows (see, e.g., [**3**; p. 57]):

(A) *Let* $v(t)$, $w(t)$ *be differentiable in* J *and* $v(0) < w(0)$, $v'(t) - f(t, v) < w'(t) - f(t, w)$ *in* J. *Then* $v < w$ *in* J.

Due to this theorem the operator $L\varphi = (\varphi' - f(t, \varphi), \varphi(0) - u_0)$ is "monotone." Using an obvious interpretation of inequalities, Theorem (A) may simply be stated as "$Lv < Lw$ implies $v < w$." Therefore the inequalities $Lv < 0$, $Lw > 0$ characterize a subfunction v and a superfunction w such that $v < u < w$ for each solution u of (1).

The existence proofs mentioned above in connection with (A) cannot be transformed to systems of ordinary differential equations; the vector analog of (A) is not true in general. It is only true if the function

$$f(t, x) = (f_1(t, x_1, \cdots, x_n), \cdots, f_n(t, x_1, \cdots, x_n))$$

is "quasimonotone increasing in x", i.e., if $f_i(t, x_1, \cdots, x_n)$ is increasing in x_j for $i \neq j$; see, e.g., [**3**; p. 83]. We remark that the existence proof given here and Perron's proof carry over to systems whose right hand side $f(t, x)$ is quasimonotone increasing in x.

(d) In 1959, when the author carried out his "Habilitation" at the University of Karlsruhe, Germany, the "*Probevorlesung*" was still part of this academic procedure. In his *Probevorlesung*, entitled *Der Existenzsatz von Peano*, the author first gave the existence proof described in this paper. Independently and about the same time H. Grunsky found another constructive existence proof [**1**] which, though being different in the technical details, has basic ideas in common with our proof.

(e) Naturally, the minimum solution of (1) may be constructed in essentially the same way. In (2) one has to replace the maximum by the minimum, and the rectangle is now given by $[t_i, t_{i+1}] \times [v_i - Mh, v_i + 3Mh]$. If $(\bar{v}_k(t))$ is the sequence constructed in this way for $h_k = 2^{-k}T$, then $(\bar{v}_k)$ turns out to be a monotone increasing sequence, and $\lim \bar{v}_k(t)$ is the minimum solution of (1). Therefore, if u is any solution of (1), then

$$\bar{v}_k \leqq \bar{v}_{k+1} \leqq u \leqq v_{k+1} \leqq v_k \qquad \text{in } J$$

for all k. In other words, our method is a numerical procedure which yields monotone sequences of upper and lower bounds.

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