APM 236 Second Midterm March 10, 2004 100 points possible

YELLOW ANSWER KEY

You may not use calculators, cell phones, or PDAs during the exam. Partial credit is possible. Please read the entire test before starting. READ EACH PROBLEM CAREFULLY.

The test ends at 7:00 pm. If even one student does not stop writing when asked, I will not curve the test.

Print your name clearly:

Print your student number clearly:

Please sign here:

Problem 1	out of 20
Problem 2	out of 10
Problem 3	out of 10
Problem 4	out of 10
Problem 5	out of 25
Problem 6	out of 25
Total	out of 100

1. (20 pt) Use the simplex method to solve the following linear programming problem:

$$\text{Maximize } -2x + y$$
$$\text{subject to}$$
$$\begin{array}{rcrcl} 2x & + & y & \geq & 3 \\ 2x & + & y & \leq & 1 \end{array}$$
$$\text{where } x, y \geq 0$$

For full credit, you have to give an optimal solution and the optimal value. You can't just give the final simplex tableau.

I'll rename x as x_1 and y as x_2.
After adding slack variables x_3 & x_4 and artificial variable y_1,

Phase 1 is:

maximize $-y_1$
subject to

$$\begin{aligned} 2x_1 + x_2 - x_3 \quad + y_1 &= 3 \\ 2x_1 + x_2 \quad + x_4 \quad &= 1 \end{aligned}$$

$x_1 \dots x_4, y_1 \geq 0$

I rewrite the objective function in terms of nonbasic variables:

maximize $2x_1 + x_2 - x_3 - 3$

first tableau of phase 1:

	x_1	x_2	x_3	x_4	y_1	
y_1	2	1	-1	0	1	3
x_4	2	1	0	1	0	1
	-2	-1	1	0	0	-3

I take x_2 entering and x_4 departing.

Second tableau of phase 1:

	x_1	x_2	x_3	x_4	y_1	
y_1	0	0	-1	-1	1	2
x_2	2	1	0	1	0	1
	0	0	1	1	0	-2

phase 1 terminated unsuccessfully!

There are no feasible solutions of the original problem and hence no optimal solutions.

2. (10 pt) Consider the linear programming problem:

$$\text{Maximize } \vec{c}^T\vec{x}$$
$$\text{subject to}$$
$$A\vec{x} \leq \vec{b}$$
$$\text{where } \vec{x} \geq \vec{0}.$$

Assume that A is an $m \times n$ matrix, $\vec{b} \in \mathbb{R}^m$, and $\vec{c} \in \mathbb{R}^n$.

Prove that if $\vec{b} \geq \vec{0}$ then there will always be feasible solutions. (*That is, show that if* $\vec{b} \geq \vec{0}$ *then you can always find* $\vec{x} \in \mathbb{R}^n$ *such that* $A\vec{x} \leq \vec{b}$ *and* $\vec{x} \geq \vec{0}$.)

I will prove there are feasible solutions by providing an example of one.

Consider $\vec{x}_0 = \vec{0}$.

$A\vec{x}_0 = A\vec{0} = \vec{0} \leq \vec{b}$ ✓ (true since $\vec{b} \geq 0$)

$\vec{x}_0 = \vec{0} \geq \vec{0}$ ✓

So $\vec{x}_0 = \vec{0}$ satisfies both constraints ($A\vec{x} \leq \vec{b}$ and $\vec{x} \geq 0$) and is therefore a feasible solution.

3. (10 pt) Consider the following simplex tableau:

	x_1	x_2	x_3	x_4	
x_3	0	-1	1	1	2
x_1	1	0	0	-1	5
	0	-2	0	2	9

This tableau arose in the process of solving a maximization problem. Prove that this maximization problem is unbounded.

The tablean encodes 2 constraints and one objective function:

maximize $2x_2 - 2x_4 + 9$

subject to

$$-x_2 + x_3 + x_4 = 2$$
$$x_1 \qquad - x_4 = 5$$

$$x_1 .. x_4 \geq 0$$

x_1 & x_3 are the basic variables.

x_2 & x_4 are nonbasic variables $\Rightarrow x_2 = x_4 = 0$

If x_2 increases from 0 to positive while $x_4 = 0$ the constraints and objective function are

$$z = 2x_2 + 9$$
$$-x_2 + x_3 = 2$$
$$x_1 = 5$$

That is as x_2 increases, x_1 remains constant, x_3 increases (and never departs), and the objective function increases. Since x_2 can get arbitrarily large, so can the objective function. $\Rightarrow$ the problem is unbounded

4. (10 pt)

a. Consider the following simplex tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_3	2	0	1	3	-1	1/2	0	1
x_2	1	1	0	-2	1	-3	0	0
x_7	-1	0	0	1	-2	1	1	1
	1	0	0	-6	-2	-2	0	4

This tableau arose as part of a maximization problem. What will you take as your entering variable and as your departing variable? What is the reason for this choice?

If X_4 enters $\Rightarrow X_3$ departs $\Rightarrow$ obj. incr by 2

if X_5 enters $\Rightarrow X_2$ departs $\Rightarrow$ obj. incr. by 0

if X_6 enters $\Rightarrow X_7$ departs $\Rightarrow$ obj incr by 2

I would take X_6 entering and X_7 departing to minimize the number of fractions in the next tableau.

b. Consider the following simplex tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_4	4	0	4	1	1	0	4	0
x_2	2	1	2	0	8	0	-2	0
x_6	6	0	2	0	-6	1	10	3
	9	0	-2	0	-4	0	-6	7

This tableau arose as part of a maximization problem. What will you take as your entering variable and as your departing variable? What is the reason for this choice?

If X_3 enters $\Rightarrow X_4$ or X_2 departs $\Rightarrow$ obj. incr. by 0

if X_5 enters $\Rightarrow X_4$ or X_2 departs $\Rightarrow$ obj. incr. by 0

if X_7 enters $\Rightarrow X_4$ departs $\Rightarrow$ obj. incr. by 0

No choice leads to an increase of the objective function. So I use Bland's rule and choose X_3 entering and X_2 departing.

5. (25 pt) Use the simplex method to solve the following linear programming problem:

$$\text{Maximize } z = 2x + y - 3$$
$$\text{subject to}$$

$$\begin{array}{rcrcl} -x & + & y & \leq & 4 \\ -x & + & y & \geq & 1 \\ -x & - & y & \leq & 1 \end{array}$$

$$\text{Where } y \geq 0$$

For full credit, you have to give an optimal solution and the optimal value. You can't just give the final simplex tableau.

x is unconstrained $\Rightarrow$ $x = x_1 - x_2$

$y \longrightarrow x_3$.

After adding slack variables x_4, x_5, x_6 and artificial variable y_1:

maximize $z = 2x_1 - 2x_2 + x_3 - 3$

subject to

$$\begin{array}{l} -x_1 + x_2 + x_3 + x_4 = 4 \\ -x_1 + x_2 + x_3 - x_5 + y_1 = 1 \\ -x_1 + x_2 - x_3 + x_6 = 1 \end{array}$$

$x_1 \ldots x_6, y_1 \geq 0$

phase 1: maximize $-y_1$

writing this interms of nonbasic variables:

maximize $-x_1 + x_2 + x_3 - x_5 - 1$

first tableau of phase 1:

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	
x_4	-1	1	1	1	0	0	0	4
y_1	-1	1	1	0	-1	0	1	1
x_6	-1	1	-1	0	0	1	0	1
	1	-1	-1	0	1	0	0	-1

x_2 entering, y_1 departing

second tableau of phase 1:

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	
x_4	0	0	0	1	1	0	-1	3
x_2	-1	1	1	0	-1	0	1	1
x_6	0	0	-2	0	1	1	-1	0
	0	0	0	0	0	0	1	0

terminated successfully.

to begin phase 2, I write the objective function

$$z = 2x_1 - 2x_2 + x_3 - 3$$

in terms of nonbasic variables.

$$z = 3x_3 - 2x_5 - 5$$

first tableau of phase 2:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	0	0	1	1	0	3
x_2	-1	1	1	0	-1	0	1
x_6	0	0	-2	0	1	1	0
	0	0	-3	0	2	0	-5

x_3 enters, x_2 departs

second tableau of phase 2:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	0	0	1	1	0	3
x_3	-1	1	1	0	-1	0	1
x_6	-2	2	0	0	-1	1	2
	-3	3	0	0	-1	0	-2

looking at the x_1 column, we see the problem is unbounded. There are feasible solutions, but no optimal solution.

6. (25 pt) Use the simplex method to solve the following linear programming problem:

$$\text{Minimize } z = x + 2y + 4$$
subject to

$$\begin{array}{rcrcl} -x & + & y & \leq & 4 \\ -x & + & y & \geq & 1 \\ -x & - & y & \leq & 1 \end{array}$$

Where $y \geq 0$

For full credit, you have to give an optimal solution and the optimal value. You can't just give the final simplex tableau.

This has the exact same set of feasible solutions as problem 5. So I'll use my phase 1 work in problem 5 and immediately start phase 2:

$$\text{maximize } -x_1 + x_2 - 2x_3 - 4$$

writing this in nonbasic variables,

$$\text{maximize } -3x_3 + x_5 - 3$$

first tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	0	0	1	1	0	3
x_2	-1	1	1	0	-1	0	1
x_6	0	0	-2	0	1	1	0
	0	0	3	0	-1	0	-3

x_5 enters, x_6 departs

second tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	0	2	1	0	-1	3
x_2	-1	1	-1	0	0	1	1
x_5	0	0	-2	0	1	1	0
	0	0	1	0	0	1	-3

terminated!

optimal solution:

$x_1 = 0 \quad x_2 = 1 \quad x_3 = 0$

$x_4 = 3 \quad x_5 = 0 \quad x_6 = 0$

opt value $= -3$

in terms of the original problem

$x = -1 \quad y = 0$

opt. value $= 3$