APM 236 Second Midterm March 10, 2004 100 points possible

WHITE ANSWER KEY

You may not use calculators, cell phones, or PDAs during the exam. Partial credit is possible. Please read the entire test before starting. READ EACH PROBLEM CAREFULLY.

The test ends at 7:00 pm. If even one student does not stop writing when asked, I will not curve the test.

Print your name clearly:

Print your student number clearly:

Please sign here:

Problem 1	out of 20
Problem 2	out of 10
Problem 3	out of 10
Problem 4	out of 10
Problem 5	out of 25
Problem 6	out of 25
Total	out of 100

1. (20 pt) Use the simplex method to solve the following linear programming problem:

$$\text{Maximize } -2x + y$$
$$\text{subject to}$$
$$\begin{array}{rcrcr} -2x & + & y & \leq & -1 \\ -2x & + & y & \geq & 1 \end{array}$$
$$\text{where } x, y \geq 0$$

For full credit, you have to give an optimal solution and the optimal value. You can't just give the final simplex tableau.

First, change 1st inequality so right hand side is ≥ 0. Also, I'll rename x as x_1 and y as x_2.

maximize $-2x_1 + x_2$
subject to

$$2x_1 - x_2 \geq 1$$
$$-2x_1 + x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

After adding slack variables x_3 & x_4 and artificial variables y_1 & y_2, I start phase 1:

maximize $-y_1 - y_2$
subject to

$$\begin{array}{rrrrrrl} 2x_1 & -x_2 & -x_3 & & +y_1 & & = 1 \\ -2x_1 & +x_2 & & -x_4 & & +y_2 & = 1 \end{array}$$
$$x_1, x_2, x_3, x_4, y_1, y_2 \geq 0$$

The basic variables are y_1 & y_2 so I write the objective function in terms of nonbasic variables:

maximize $-x_3 - x_4 - 2$

the first simplex tableau of phase 1 is then:

	x_1	x_2	x_3	x_4	y_1	y_2	
y_1	2	-1	-1	0	1	0	1
y_2	-2	1	0	-1	0	1	1
	0	0	1	1	0	0	-2

no pivoting is necessary... phase 1 has terminated unsuccessfully. There are no feasible solutions and hence no optimal solution.

2. (10 pt) Consider the linear programming problem:

$$\text{Maximize } \vec{c}^T\vec{x}$$
$$\text{subject to}$$
$$A\vec{x} \leq \vec{b}$$
$$\text{where } \vec{x} \geq \vec{0}.$$

Assume that A is an $m \times n$ matrix, $\vec{b} \in \mathbb{R}^m$, and $\vec{c} \in \mathbb{R}^n$.

Prove that if $\vec{b} \geq \vec{0}$ then there will always be feasible solutions. (*That is, show that if* $\vec{b} \geq \vec{0}$ *then you can always find* $\vec{x} \in \mathbb{R}^n$ *such that* $A\vec{x} \leq \vec{b}$ *and* $\vec{x} \geq \vec{0}$.)

I will prove there are feasible solutions by providing an example of one.

Consider $\vec{x}_0 = \vec{0}$.

$A\vec{x}_0 = A\vec{0} = \vec{0} \leq \vec{b}$ ✓ (true since $\vec{b} \geq 0$)

$\vec{x}_0 = \vec{0} \geq \vec{0}$ ✓

So $\vec{x}_0 = \vec{0}$ satisfies both constraints ($A\vec{x} \leq \vec{b}$ and $\vec{x} \geq 0$) and is therefore a feasible solution.

3. (10 pt) Consider the following simplex tableau:

	x_1	x_2	x_3	x_4	
x_4	0	0	2	1	3
x_2	-2	1	9	0	4
	-1	0	1	0	5

This tableau arose in the process of solving a maximization problem. Prove that this maximization problem is unbounded.

The tableau encodes two constraints and one objective function:

maximize $x_1 - x_3 + 5$

subject to

$$2x_3 + x_4 = 3$$

$$-2x_1 + x_2 + 9x_3 = 4$$

$$x_1, \dots x_4 \geq 0$$

x_2 & x_4 are basic variables.

x_1 & x_3 are nonbasic variables $\Rightarrow x_1 = x_3 = 0$

If x_1 increases from 0 while $x_3 = 0$ the constraints & objective function are

$$z = x_1 + 5$$

$$x_4 = 3$$

$$-2x_1 + x_2 = 4$$

That is, as x_1 gets larger and larger, z increases and x_4 remains constant and x_2 increases. No basic variable departs and the objective function can be arbitrarily large.

This proves the problem is unbounded.

4. (10 pt)

a. Consider the following simplex tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_3	-1	0	1	4	-2	1	0	0
x_2	2	1	0	2	1	3	0	0
x_7	-1	0	0	4	6	1	1	10
	8	0	0	-2	-1	-10	0	-4

This tableau arose as part of a maximization problem. What will you take as your entering variable and as your departing variable? What is the reason for this choice?

If x_4 enters then either x_3 or x_2 can depart. The objective function increases by 0.

If x_5 enters then x_2 departs and the objective function increases by 0.

If x_6 enters then either x_3 or x_2 can depart. The objective function increases by 0.

Since no entering variable leads to an increase, we have to use Bland's rule: x_4 enters, x_2 departs.

b. Consider the following simplex tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_4	-4	0	3	1	-1	0	-3	0
x_2	5	1	-1	0	-4	0	2	1
x_6	1	0	-4	0	1	1	3	3
	2	0	-3	0	-1	0	-6	2

This tableau arose as part of a maximization problem. What will you take as your entering variable and as your departing variable? What is the reason for this choice?

If x_3 enters $\Rightarrow$ x_4 departs $\Rightarrow$ 0 incr. of obj. funct.

If x_5 enters $\Rightarrow$ x_6 departs $\Rightarrow$ incr. of 3 of obj. funct.

If x_7 enters $\Rightarrow$ x_2 departs $\Rightarrow$ incr of 3 of obj. funct.

I would take x_5 entering and x_6 departing. This gives the greatest increase of the objective function and avoids nasty fractions in the subsequent tableau.

5. (25 pt) Use the simplex method to solve the following linear programming problem:

$$\text{Maximize } z = -x + 2y + 1$$
$$\text{subject to}$$
$$\begin{array}{rcrcl} x & - & y & \leq & 4 \\ x & - & y & \geq & 1 \\ -x & - & y & \leq & 1 \end{array}$$
$$\text{Where } x \geq 0$$

For full credit, you have to give an optimal solution and the optimal value. You can't just give the final simplex tableau.

y is unconstrained $\Rightarrow$ I'll replace y by $x_2 - x_3$

I'll replace x by x_1.

maximize $-x_1 + 2x_2 - 2x_3 + 1$

subject to

$$\begin{array}{rl} x_1 - x_2 + x_3 & \leq 4 \\ x_1 - x_2 + x_3 & \geq 1 \\ -x_1 - x_2 + x_3 & \leq 1 \end{array}$$

$x_1, x_2, x_3 \geq 0$

Adding slack variables x_4, x_5, x_6 and artificial variable y_1, phase 1 is

maximize $-y_1$

subject to

$$\begin{array}{rl} x_1 - x_2 + x_3 + x_4 & = 4 \\ x_1 - x_2 + x_3 - x_5 + y_1 & = 1 \\ -x_1 - x_2 + x_3 + x_6 & = 1 \end{array}$$

$x_1 \ldots x_6, y_1 \geq 0$

I write the objective function in terms of nonbasic variables:

maximize $x_1 - x_2 + x_3 - x_5 - 1$

first tableau of phase 1:

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	
x_4	1	−1	1	1	0	0	0	4
y_1	1	−1	1	0	−1	0	1	1
x_6	−1	−1	1	0	0	1	0	1
	−1	1	−1	0	1	0	0	−1

I take x_3 entering & y_1 departing

second tableau of phase 1:

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	
x_4	0	0	0	1	1	0	−1	3
x_3	1	−1	1	0	−1	0	1	1
x_6	−2	0	0	0	1	1	−1	0
	0	0	0	0	0	0	1	0

phase 1 has terminated successfully!

to start phase 2, I write the objective function

$$-x_1 + 2x_2 - 2x_3 + 1$$ in terms of non basic variables.

eq 2 $\Rightarrow$ $x_3 = -x_1 + x_2 + x_5 + 1$

$\Rightarrow$ maximize $x_1 - 2x_5 - 1$

the first tableau of phase 2 is:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	0	0	1	1	0	3
x_3	1	-1	1	0	-1	0	1
x_6	-2	0	0	0	1	1	0
	-1	0	0	0	2	0	-1

x_1 enters, x_3 departs

the second tableau of phase 2 is

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	0	0	1	1	0	3
x_1	1	-1	1	0	-1	0	1
x_6	0	-2	2	0	-1	1	2
	0	-1	1	0	1	0	0

the problem is unbounded! there are feasible solutions, but no optimal solution

6. (25 pt) Use the simplex method to solve the following linear programming problem:

$$\text{Minimize } z = 2x + y + 5$$
$$\text{subject to}$$
$$\begin{array}{rcrcl} x & - & y & \leq & 4 \\ x & - & y & \geq & 1 \\ -x & - & y & \leq & 1 \end{array}$$
$$\text{Where } x \geq 0$$

For full credit, you have to give an optimal solution and the optimal value. You can't just give the final simplex tableau.

This has the exact same set of feasible solutions as problem 5, so I'll use my phase 1 work in problem 5 and immediately start phase 2:

$$\text{maximize } -2X_1 - X_2 + X_3 - 5$$

Writing this in nonbasic variables,

$$\text{maximize } -3X_1 + X_5 - 4$$

first tableau:

	X_1	X_2	X_3	X_4	X_5	X_6	
X_4	0	0	0	1	1	0	3
X_3	1	-1	1	0	-1	0	1
X_6	-2	0	0	0	1	1	0
	3	0	0	0	-1	0	-4

X_5 enters, X_6 departs.

second tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	2	0	0	1	0	-1	3
x_3	-1	-1	1	0	0	1	1
x_5	-2	0	0	0	1	1	0
	1	0	0	0	0	1	-4

terminates! optimal solution

$x_1 = 0 \quad x_2 = 0 \quad x_3 = 1$

$x_4 = 3 \quad x_5 = 0 \quad x_6 = 0$ opt. value = -4

in terms of the original problem:

$x = 0 \quad y = -1$

opt value = 4