APM 236 First Midterm (February 4, 2004) 100 points possible

You may not use calculators, cell phones, or PDAs during the exam. Partial credit is possible. Please read the entire test over before starting.

The test ends at 7:00 pm. If even one student does not stop writing when asked, I will not curve the test.

Print your name clearly:

White answer Key

Please sign here:

Problem 1	out of 20
Problem 2	out of 20
Problem 3	out of 20
Problem 4	out of 20
Problem 5	out of 20
TOTAL	out of 100

1. (20 points) An advertising manager is assigned the task of putting together an advertising campaign for a client. The client has set the objectives of getting at least 160 million exposures, with at least 60 million of those exposures being persons with income of at least \$8000 per year and at least 80 million in the 18 to 40-year-old-age groups. Market studies indicate that an ad in a certain magazine will be seen by 8 million people of whom 3 million will be in the targeted income group and 4 million will be in the targeted age group. Each magazine ad will cost \$40,000. An ad on television will be seen by 40 million people of whom 10 million will be in the targeted income group and 10 million will be in the targeted age group. Each television ad will cost \$200,000. How can the objectives of this advertising campaign be achieved at least cost?

Source: William J. Baumol, "Economic Theory and Operations Analysis", 1972, page 98

Please write this word problem as a linear programming problem.

Let x = # of magazine ads
y = # of tv ads.

# of exposures in right age group (in millions)
$= 4x + 10y$

# of exposures in right income bracket
$= 3x + 10y$

total # of exposures
$= 8x + 40y$

cost $= 40x + 200y$ (in thousands)

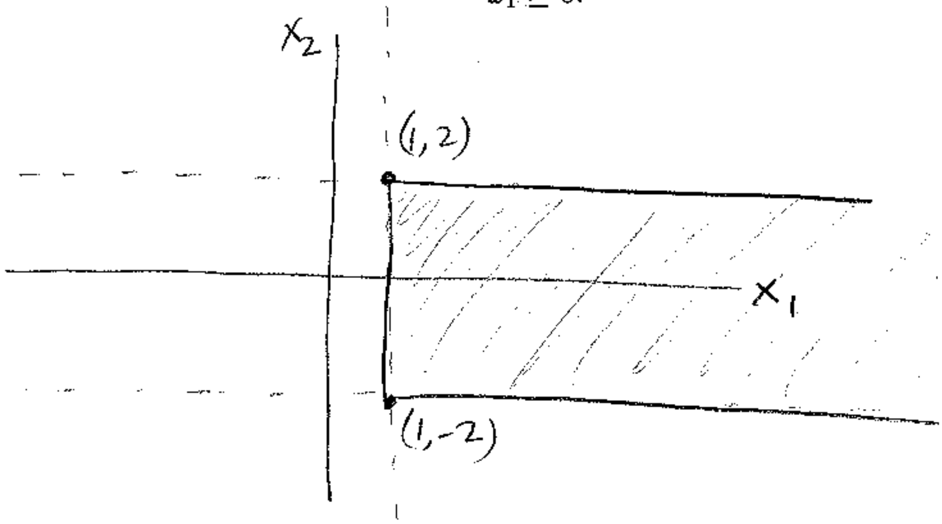
$$\begin{cases} \text{minimize } 4x + 200y \\ \text{subject to} \\ \quad 8x + 40y \geq 160 \\ \quad 3x + 10y \geq 60 \\ \quad 4x + 10y \geq 80 \\ x, y \geq 0 \end{cases}$$

2. For each of the following four problems, use graphical methods to find the optimal solution(s) if any.

a. (5 points)

$$\begin{array}{lrcl} \text{Maximize} & x_1 + x_2 & & \\ \text{subject to} & & & \\ & x_1 & \geq & 1 \\ & x_2 & \leq & 2 \\ & -x_2 & \leq & 2 \\ & x_1 \geq 0. & & \end{array}$$

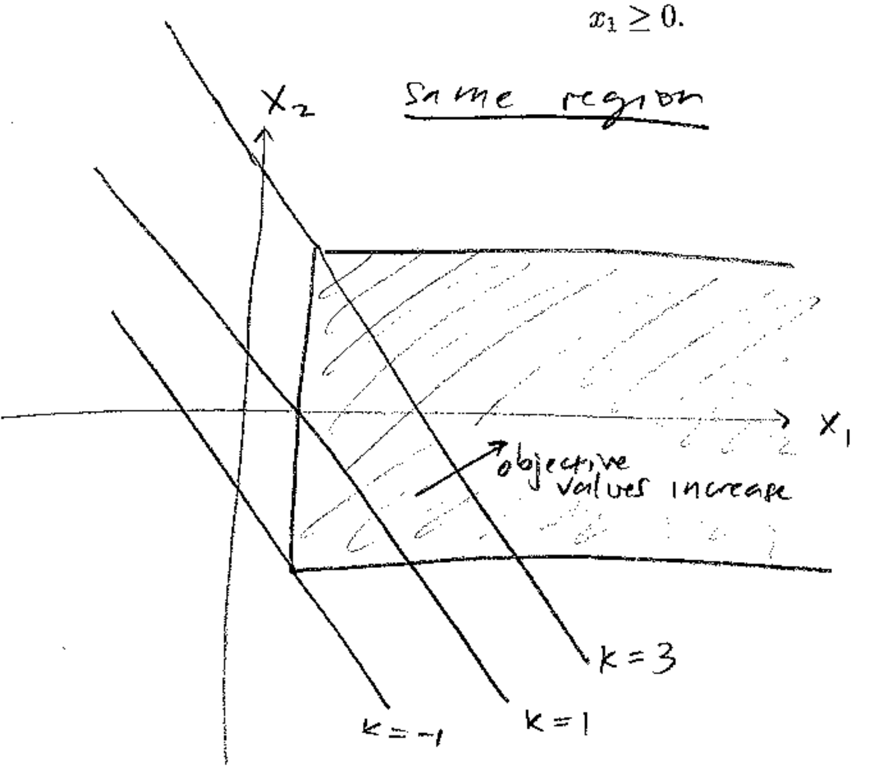
$\Rightarrow x_2 \geq -2$



no maximum exists. consider objective function on top face: $(x,2)$ where $x \geq 1$. Then obj. function equals
$x+2$
and this has no max.

b.

$$\begin{array}{lrcl} \text{Minimize} & x_1 + x_2 & & \\ \text{subject to} & & & \\ & x_1 & \geq & 1 \\ & x_2 & \leq & 2 \\ & -x_2 & \leq & 2 \\ & x_1 \geq 0. & & \end{array}$$



level sets:
$x_1 + x_2 = k$

optimal solution at $(1,-2)$
optimal value $= -1$

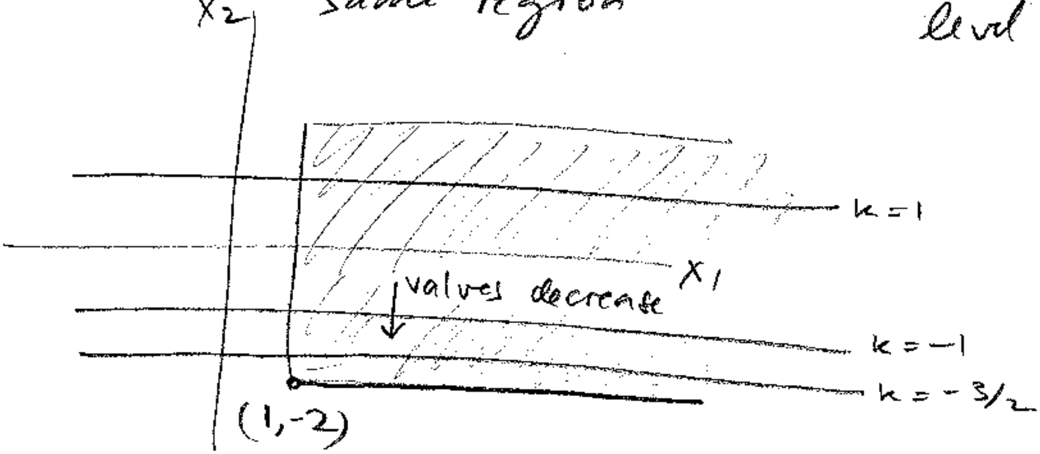
c.

$$
\begin{array}{ll}
\text{Minimize} & x_2 \\
\text{subject to} & \\
& \begin{array}{rcl} x_1 \quad\quad & \geq & 1 \\ x_2 & \leq & 2 \\ -x_2 & \leq & 2 \end{array} \\
x_1 \geq 0. &
\end{array}
$$

same region

level sets:

$x_2 = k$



infinitely many optimal solutions. occur on $x_1 \geq 1$, $x_2 = -2$

Minimal value $= -2$

d.

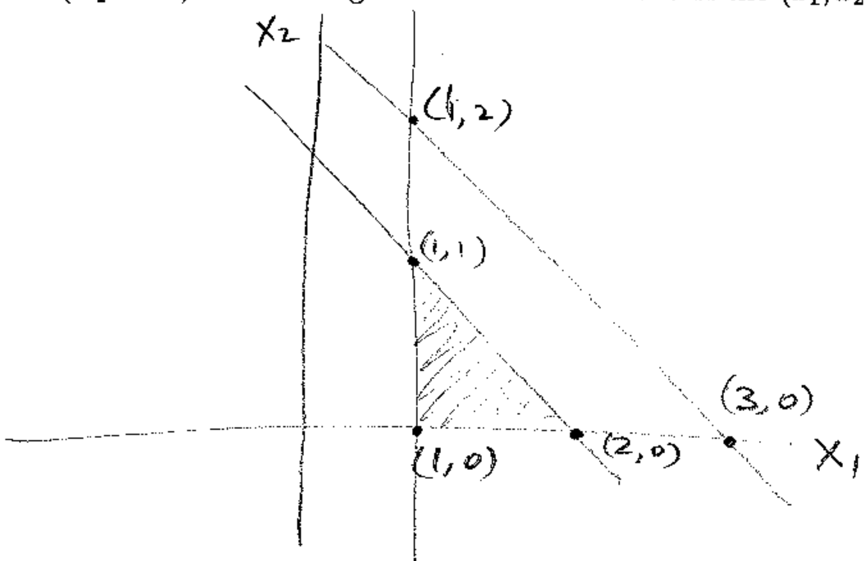
$$
\begin{array}{ll}
\text{Minimize} & x_2 \\
\text{subject to} & \\
& \begin{array}{rcl} x_1 \quad\quad & \leq & 1 \\ x_2 & \leq & 0 \\ -x_2 & \leq & -2 \end{array} \quad \rightarrow \quad x_2 \geq 2 \\
x_1 \geq 0. &
\end{array}
$$

there are no feasible solutions because it is impossible to have $x_2 \leq 0$ and $x_2 \geq 2$.

3. Consider the following linear programming problem:

$$\begin{aligned} \text{Minimize} \quad & -2x_1 - 2x_2 \\ \text{subject to} \quad & \\ x_1 &\geq 1 \\ x_1 + x_2 &\leq 2 \\ x_1 + x_2 &\leq 3 \\ x_1, x_2 \geq 0. & \end{aligned}$$

a. (5 points) Plot the region of feasible solutions in the (x_1, x_2) plane.



b. (1 points) List all extreme points of the set of feasible solutions.

(1, 0) , (2, 0) , (1, 1)

c. (4 points) Find the optimal solution(s) of the linear programming problem.

at (1,0) obj function = -2

at (2,0) " " = -4

at (1,1) " " = -4

infinitely many optimal solutions. occur on the segment $X_1+X_2=2$, $1 \leq X_1 \leq 2$.

minimal value = -4

d. (2 points) Write the linear programming problem in canonical form.

minimize $-2x_1 - 2x_2$

subject to

$$
\begin{aligned}
x_1 \quad\quad - x_3 \quad\quad\quad &= 1 \\
x_1 + x_2 \quad\quad + x_4 \quad &= 2 \\
x_1 + x_2 \quad\quad\quad + x_5 &= 3
\end{aligned}
$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

e. (1 points) The feasible solutions of the linear programming problem in part d can be written as "The set of $\vec{x}$ that satisfy $A\vec{x} = \vec{b}$ and $\vec{x} \geq \vec{0}$". What is the matrix A? What is the vector $\vec{b}$?

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

f. (3 points) For each extreme point (x_1, x_2) you found in part b, what is the corresponding extreme point of the linear programming problem that you found in part d?

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

g. (4 points) Use the first, second, and fourth columns of A to find a basic solution $\vec{x}$. Is your basic solution a basic feasible solution? Looking at your plot in part a, to what does your basic solution correspond?

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Gaussian elimination.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right)$$

$$\downarrow$$

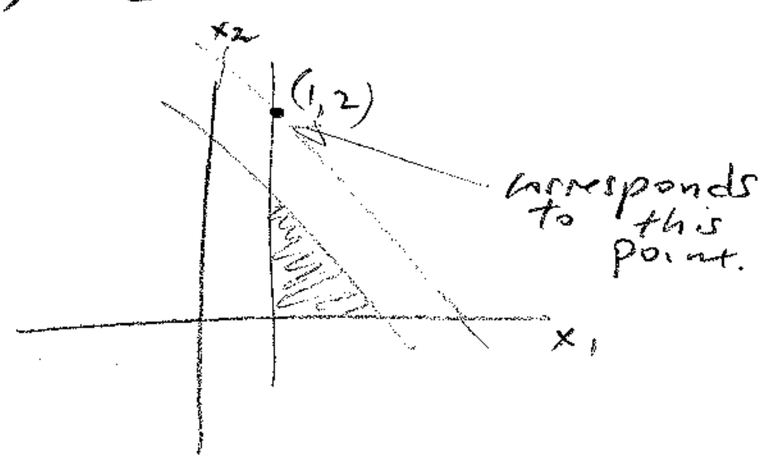
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

eq 3 $\Rightarrow$ $-x_4 = 1$ $\Rightarrow$ $x_4 = -1$

eq 2 $\Rightarrow$ $x_2 + x_4 = 1$ $\Rightarrow$ $x_2 = 2$

eq 1 $\Rightarrow$ $x_1 = 1$

basic soln: $\begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

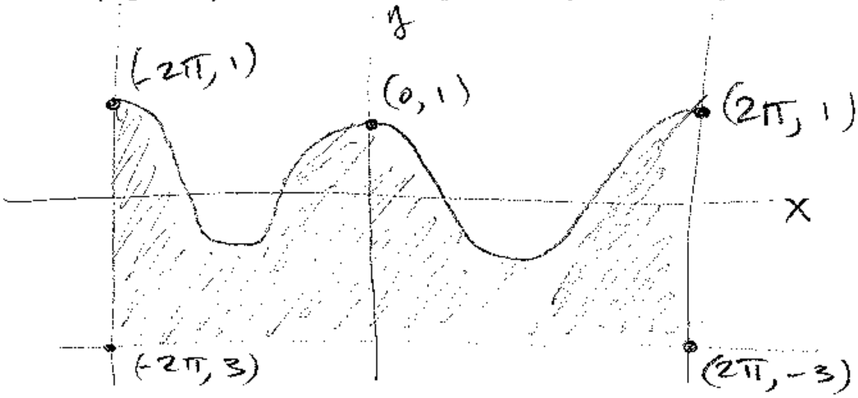


not feasible

4. a. Consider the set of points (x, y) in $\mathbb{R}^2$ that satisfy the following four inequalities:

$$\begin{cases} x & \geq -2\pi \\ x & \leq 2\pi \\ y & \leq \cos(x) \\ y & \geq -3 \end{cases}$$

(3 points) Plot this set of points. By visual inspection, is this a convex set in $\mathbb{R}^2$?



is not convex.

(7 points) Prove or disprove that this set is convex. *Recall that "a set in $\mathbb{R}^2$ is convex if (x_0, y_0) and (x_1, y_1) are in the set then $(tx_0 + (1-t)x_1, ty_0 + (1-t)y_1)$ is in the set for all $0 < t < 1$."*

proof that it's not convex:

$(0, 1)$ is in the region

$(2\pi, 1)$ is in the region

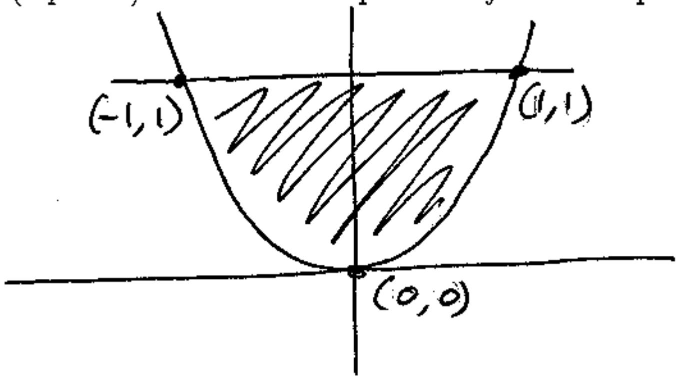
$(\pi, 1)$ is the midpoint of the segment connecting $(0,1)$ to $(2\pi, 1)$. But $(\pi, 1)$ is not in the region since $1 \not\leq \cos(\pi) = -1$

This shows the region is not convex.

b. Consider the set of points (x, y) in $\mathbb{R}^2$ that satisfy the following two inequalities:

$$\begin{cases} y & \geq x^2 \\ y & \leq 1 \end{cases}$$

(2 points) Plot this set of points. By visual inspection, is this a convex set in $\mathbb{R}^2$?



is convex.

(8 points) Prove or disprove that this set is convex. *Recall that "a set in $\mathbb{R}^2$ is convex if (x_0, y_0) and (x_1, y_1) are in the set then $(tx_0 + (1-t)x_1, ty_0 + (1-t)y_1)$ is in the set for all $0 < t < 1$."*

proof that the region is convex:

Assume (x_0, y_0) and (x_1, y_1) are in the region. I need to show that for any $0<t<1$, $(tx_0+(1-t)x_1, ty_0+(1-t)y_1)$ is in the region.

Step 1: show $ty_0+(1-t)y_1 \leq 1$.

Since $t>0$ and $1-t>0$ and $y_0 \leq 1$ and $y_1 \leq 1$ (by assumption) I know $ty_0 \leq t$ and $(1-t)y_1 \leq 1-t$.

Hence $ty_0+(1-t)y_1 \leq t+1-t = 1$, as desired.

Step 2: show $ty_0+(1-t)y_1 \geq [tx_0+(1-t)x_1]^2$

Since $t>0$ and $1-t>0$ and $y_0 \geq x_0^2$ and $y_1 \geq x_1^2$, I know $ty_0 \geq tx_0^2$ and $(1-t)y_1 \geq (1-t)x_1^2$.

I want to use this to show:

$$ty_0+(1-t)y_1 \geq [tx_0+(1-t)x_1]^2$$

A quick calculation shows

$$[tx_0+(1-t)x_1]^2 = tx_0^2+(1-t)x_1^2 - t(1-t)(x_0-x_1)^2$$
$$\leq tx_0^2+(1-t)x_1^2 \leq ty_0+(1-t)y_1$$

as desired. //

5. a. (10 points) Write the following linear programming problem as a linear programming problem in **canonical form**.

Once you have done this, what is your matrix A and your vectors $\vec{c}$ and $\vec{b}$ so that the problem becomes "Maximize $\vec{c} \cdot \vec{x}$ subject to $A\vec{x} = \vec{b}$ and $\vec{x} \geq \vec{0}$?"

$$\begin{array}{lcccl}
\text{Minimize} & & x_1 - 5x_2 & & \\
\text{subject to} & & & & \\
-x_1 & + & 4x_2 & = & 10 \\
2x_1 & + & x_2 & \leq & 2 \\
x_1 & - & 7x_2 & \geq & -3 \\
x_1, x_2 \geq 0. & & & &
\end{array}$$

$$\left\{\begin{array}{l}
\text{maximize} \quad -x_1 + 5x_2 \\
\text{subject to} \\
\quad -x_1 + 4x_2 = 10 \\
\quad 2x_1 + x_2 + x_3 = 2 \\
\quad x_1 - 7x_2 - x_4 = -3 \\
x_1, x_2, x_3, x_4 \geq 0
\end{array}\right.$$

$$A = \begin{pmatrix} -1 & 4 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -7 & 0 & -1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 10 \\ 2 \\ -3 \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} -1 \\ 5 \\ 0 \\ 0 \end{pmatrix}$$

b. (10 points) Write the following linear programming problem in **standard form**.

Once you have done this, what is your matrix A and your vectors $\vec{c}$ and $\vec{b}$ so that the problem becomes "Maximize $\vec{c} \cdot \vec{x}$ subject to $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq \vec{0}$?"

$$\begin{array}{llcll} \text{Maximize} & & -x_1 + 10x_2 & & \\ \text{subject to} & & & & \\ -x_1 & + & 2x_2 & = & 4 \\ x_1 & + & 3x_2 & \geq & 6 \\ x_1 \geq 0. & & & & \end{array}$$

x_2 is unconstrained.
introduce $x_2 = u - v$ where $u, v \geq 0$

maximize $-x_1 + 10u - 10v$
subject to

$$-x_1 + 2u - 2v \leq 4$$

$$x_1 - 2u + 2v \leq -4$$

$$-x_1 - 3u + 3v \leq -6$$

$$x_1, u, v \geq 0$$

$\square = \triangle$
$\square \leq \triangle$
$\square \geq \triangle$
$-\square \leq -\triangle$