APM 236 First Midterm (February 4, 2004) 100 points possible

You may not use calculators, cell phones, or PDAs during the exam. Partial credit is possible. Please read the entire test over before starting.

The test ends at 7:00 pm. If even one student does not stop writing when asked, I will not curve the test.

Print your name clearly: yellow answer key

Please sign here:

Problem 1	out of 20
Problem 2	out of 20
Problem 3	out of 20
Problem 4	out of 20
Problem 5	out of 20
TOTAL	out of 100

1. a. (10 points) Write the following linear programming problem as a linear programming problem in **canonical form**.

Once you have done this, what is your matrix A and your vectors $\vec{c}$ and $\vec{b}$ so that the problem becomes "Maximize $\vec{c}\cdot\vec{x}$ subject to $A\vec{x}=\vec{b}$ and $\vec{x}\geq\vec{0}$?"

$$\begin{array}{lcccc}
\text{Minimize} & & -x_1+2x_2 & & \\
\text{subject to} & & & & \\
x_1 & + & x_2 & \leq & 3 \\
2x_1 & - & 3x_2 & \geq & 7 \\
x_1 & - & x_2 & = & 8 \\
x_1, x_2 \geq 0. & & & &
\end{array}$$

$$\left\{\begin{array}{l}
\text{maximize} \quad x_1 - 2x_2 \\
\text{subject to} \\
\quad x_1 + x_2 + x_3 = 3 \\
\quad 2x_1 - 3x_2 - x_4 = 7 \\
\quad x_1 - x_2 = 8 \\
x_1, x_2, x_3, x_4 \geq 0
\end{array}\right.$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & -3 & 0 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}$$

b. (10 points) Write the following linear programming problem in **standard form**.

Once you have done this, what is your matrix A and your vectors $\vec{c}$ and $\vec{b}$ so that the problem becomes "Maximize $\vec{c} \cdot \vec{x}$ subject to $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq \vec{0}$?"

$$\begin{array}{lcccc} \text{Maximize} & & -2x_1 + 7x_2 & & \\ \text{subject to} & & & & \\ x_1 & - & 2x_2 & \geq & 9 \\ x_1 & + & x_2 & = & 3 \\ x_2 \geq 0. & & & & \end{array}$$

no constraint on x_1. Introduce $x_1 = u - v$

$$\begin{cases} \text{maximize} \quad -2u + 2v + 7x_2 \\ \text{subject to} \\ \quad -u + v + 2x_2 \leq -9 \\ \quad u - v + x_2 \leq 3 \\ \quad -u + v - x_2 \leq -3 \\ u, v, x_2 \geq 0 \end{cases}$$

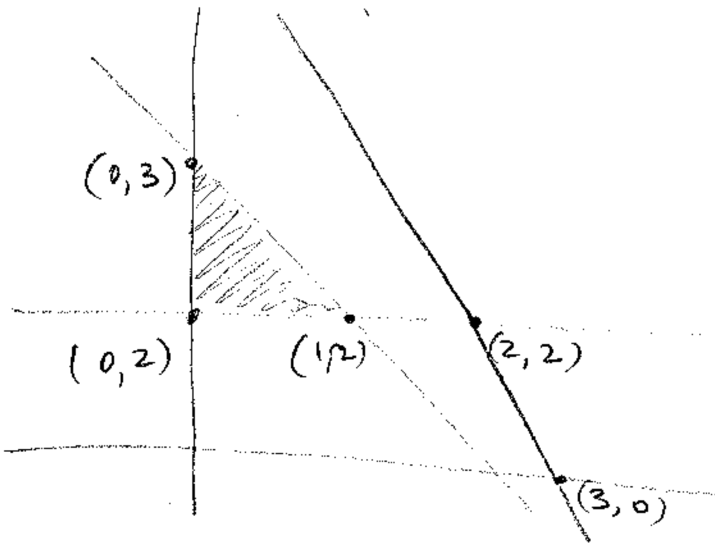
$$A = \begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} -9 \\ 3 \\ -3 \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} -2 \\ 2 \\ 7 \end{pmatrix}$$

2. Consider the following linear programming problem:

$$\begin{aligned} &\text{Minimize } x_2 \\ &\text{subject to} \\ &2x_1 + x_2 \leq 6 \\ &x_1 + x_2 \leq 3 \\ &x_2 \geq 2 \\ &x_1, x_2 \geq 0. \end{aligned}$$

a. (5 points) Plot the region of feasible solutions in the (x_1, x_2) plane.



b. (1 points) List all extreme points of the set of feasible solutions.

(0, 3), (0, 2), (1, 2)

c. (4 points) Find the optimal solution(s) of the linear programming problem.

objective function at (0,3) = 3

" at (0,2) = 2

" at (1,2) = 2

infinitely many optimal solutions

on $x_2 = 2$ $\quad 0 \leq x_1 \leq 1$

d. (2 points) Write the linear programming problem in canonical form.

$$\begin{aligned}
&\text{maximize } -x_2\\
&\text{subject to}\\
&\quad 2x_1 + x_2 + x_3 = 6\\
&\quad x_1 + x_2 + x_4 = 3\\
&\quad x_2 - x_5 = 2\\
&\quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}$$

e. (1 points) The feasible solutions of the linear programming problem in part d can be written as "The set of $\vec{x}$ that satisfy $A\vec{x} = \vec{b}$ and $\vec{x} \geq \vec{0}$". What is the matrix A? What is the vector $\vec{b}$?

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$$

f. (3 points) For each extreme point (x_1, x_2) you found in part b, what is the corresponding extreme point of the linear programming problem that you found in part d?

$$\begin{pmatrix} 0 \\ 3 \\ 3 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 4 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

g. (4 points) Use the first, second, and fourth columns of A to find a basic solution $\vec{x}$. Is your basic solution a basic feasible solution? Looking at your plot in part a, to what does your basic solution correspond?

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$$

Gaussian elimination

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 6 \\ 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 3 \\ 0 & 1/2 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{array}\right)$$

$$\downarrow$$

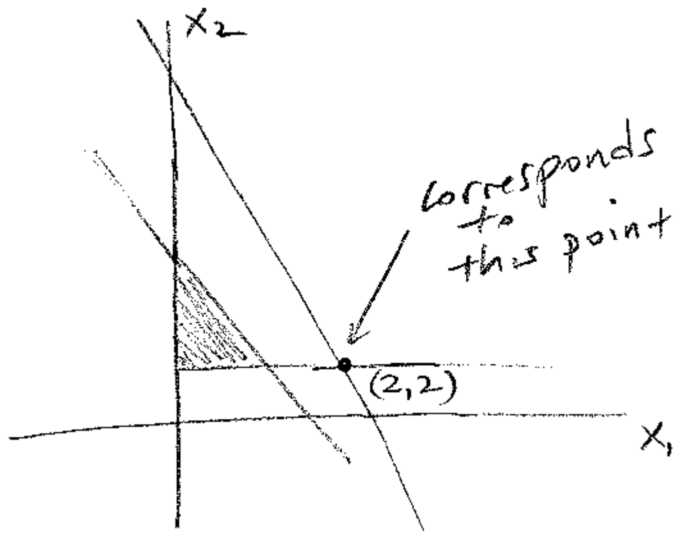
$$\left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 2 \end{array}\right)$$

eq 3 $\Rightarrow -2x_4 = 2 \Rightarrow x_4 = -1$

eq 2 $\Rightarrow x_2 + 2x_4 = 0 \Rightarrow x_2 = 2$

eq 1 $\Rightarrow x_1 + 1/2\, x_2 = 3 \Rightarrow x_1 = 2$

basic solution $\begin{pmatrix} 2 \\ 2 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

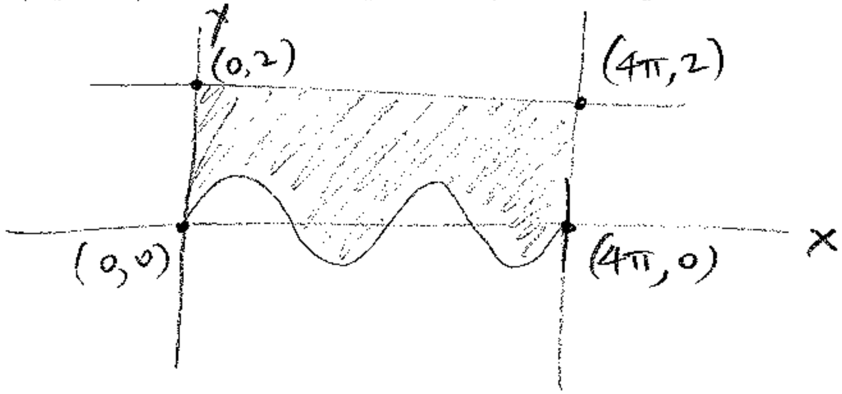


is not feasible.

3. a. Consider the set of points (x, y) in $\mathbb{R}^2$ that satisfy the following four inequalities:

$$\begin{cases} x & \geq 0 \\ x & \leq 4\pi \\ y & \geq \sin(x) \\ y & \leq 2 \end{cases}$$

(3 points) Plot this set of points. By visual inspection, is this a convex set in $\mathbb{R}^2$?



not convex

(7 points) Prove or disprove that this set is convex. *Recall that "a set in $\mathbb{R}^2$ is convex if (x_0, y_0) and (x_1, y_1) are in the set then $(tx_0 + (1-t)x_1, ty_0 + (1-t)y_1)$ is in the set for all $0 < t < 1$."*

$(0,0)$ is in the region

$(\pi, 0)$ is in the region

$(\frac{\pi}{2}, 0)$ is the midpoint of the segment connecting the two.

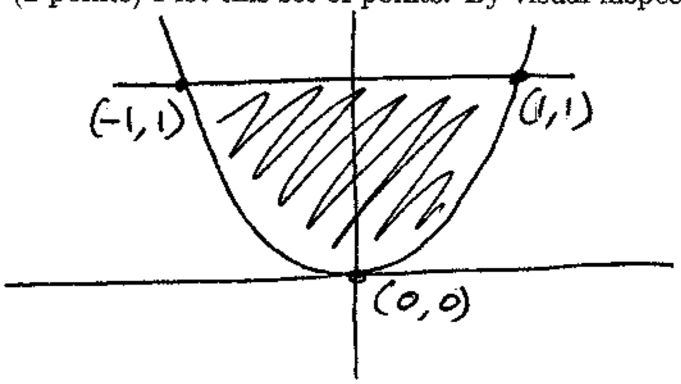
It's not in the region because

$0 \not\geq \sin(\pi/2) = 1$.

b. Consider the set of points (x, y) in $\mathbb{R}^2$ that satisfy the following two inequalities:

$$\begin{cases} y & \geq x^2 \\ y & \leq 1 \end{cases}$$

(2 points) Plot this set of points. By visual inspection, is this a convex set in $\mathbb{R}^2$?



is convex.

(8 points) Prove or disprove that this set is convex. *Recall that "a set in $\mathbb{R}^2$ is convex if (x_0, y_0) and (x_1, y_1) are in the set then $(tx_0 + (1-t)x_1, ty_0 + (1-t)y_1)$ is in the set for all $0 < t < 1$."*

proof that the region is convex:

Assume (x_0, y_0) and (x_1, y_1) are in the region. I need to show that for any $0<t<1$, $(tx_0+(1-t)x_1,\ ty_0+(1-t)y_1)$ is in the region.

Step 1: show $ty_0+(1-t)y_1 \leq 1$.

Since $t>0$ and $1-t>0$ and $y_0 \leq 1$ and $y_1 \leq 1$ (by assumption) I know $ty_0 \leq t$ and $(1-t)y_1 \leq 1-t$.

Hence $ty_0+(1-t)y_1 \leq t + 1 - t = 1$, as desired.

Step 2: show $ty_0+(1-t)y_1 \geq [tx_0+(1-t)x_1]^2$

Since $t>0$ and $1-t>0$ and $y_0 \geq x_0^2$ and $y_1 \geq x_1^2$, I know $ty_0 \geq tx_0^2$ and $(1-t)y_1 \geq (1-t)x_1^2$.

I want to use this to show:

$$ty_0+(1-t)y_1 \geq [tx_0+(1-t)x_1]^2$$

A quick calculation shows

$$[tx_0+(1-t)x_1]^2 = tx_0^2+(1-t)x_1^2 - t(1-t)(x_0-x_1)^2$$
$$\leq tx_0^2+(1-t)x_1^2 \leq ty_0+(1-t)y_1$$

as desired. //

4. (20 points) A welfare recipient decides to see if his knowledge of linear programming can help him with his financial problems. Peanut butter costs 20 cents an ounce and contains 1 unit of carbohydrate and 1 unit of protein. A small loaf of bread costs 12 cents and contains 1 unit of carbohydrate and no protein. A cup of milk costs 16 cents and contains 1 unit of protein and no carbohydrate. He estimates that he needs .9 units of carbohydrate and .6 units of protein per day. How can he get at least that much nutrient from these foods at least cost?

Source: Harold W. Kuhn, unpublished class notes.

Please write this word problem as a linear programming problem.

x = ounces of peanut butter

y = loaves of bread

z = cups of milk.

units of carbohydrates

$= x + y + 0z \geq .9$

units of protein

$= x + 0y + 1z \geq .6$

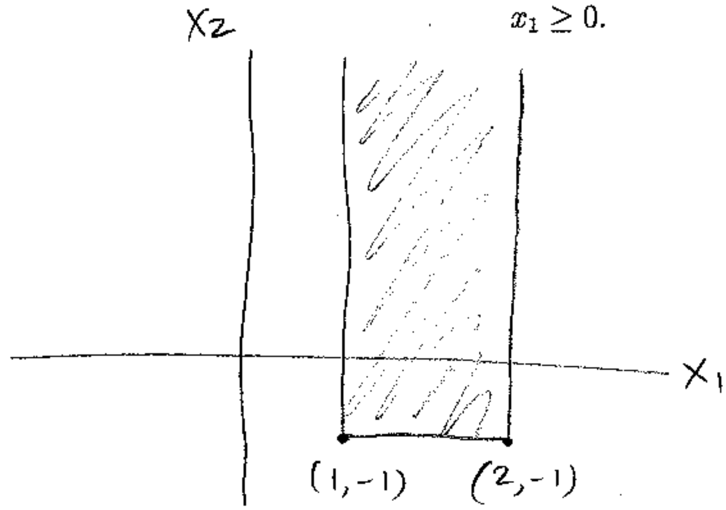
cost $= 20x + 12y + 16z$

$$\begin{cases} \text{minimize } 20x + 12y + 16z \\ \text{subject to} \\ \quad x + y \geq .9 \\ \quad x + z \geq .6 \\ x, y, z \geq 0 \end{cases}$$

5. For each of the following four problems, use graphical methods to find the optimal solution(s) if any.

a. (5 points)

$$\begin{array}{rrcrl}
\text{Maximize} & x_1 + x_2 & & & \\
\text{subject to} & & & & \\
& -x_2 & \leq & 1 & \rightarrow \quad x_2 \geq -1 \\
x_1 & & \leq & 2 & \\
-x_1 & & \leq & -1 & \rightarrow \quad x_1 \geq 1 \\
x_1 \geq 0. & & & &
\end{array}$$

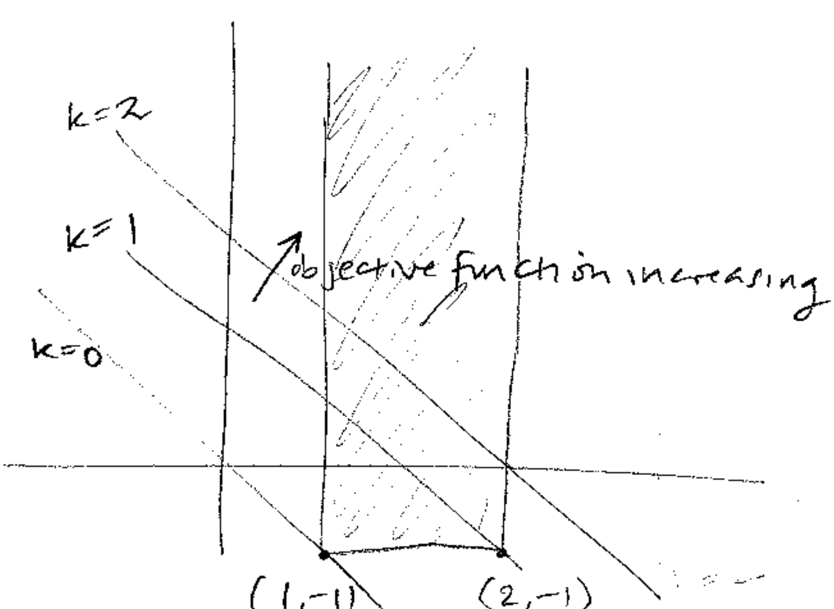


no maximum value. evaluate objective function along $x_1 = 1 \Rightarrow$ objective function $= 1 + x_2$ where $x_2 \geq -1$. This value can go to infinity

b. (5 points)

$$\begin{array}{rrcr}
\text{Minimize} & x_1 + x_2 & & \\
\text{subject to} & & & \\
& -x_2 & \leq & 1 \\
x_1 & & \leq & 2 \\
-x_1 & & \leq & -1 \\
x_1 \geq 0. & & &
\end{array}$$

Same region



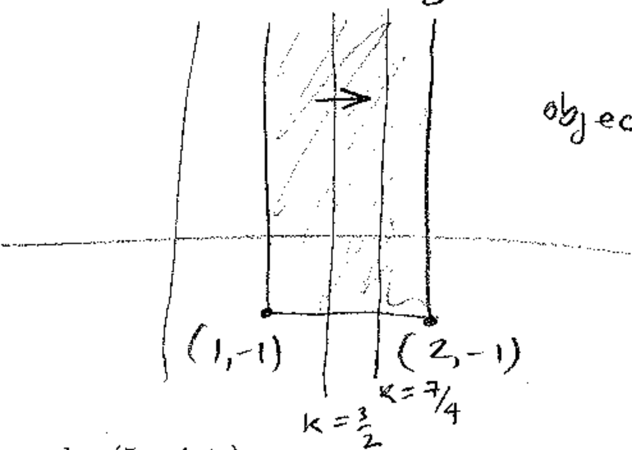
draw level sets $x_1 + x_2 = k$

Minimum achieved at $(1,-1)$. minimum value $= 0$.

c. (5 points)

$$\begin{array}{lrcl} \text{Maximize} & x_1 & & \\ \text{subject to} & & & \\ & -x_2 & \leq & 1 \\ & x_1 & \leq & 2 \\ & -x_1 & \leq & -1 \\ & x_1 \geq 0. & & \end{array}$$

same region.



draw level sets $x_1 = k$

objective function increases as level sets move to the right.

maximum value = 2 achieved at infinitely many points:

$x_1 = 2,\ x_2 \geq -1$

d. (5 points)

$$\begin{array}{lrcl} \text{Maximize} & x_1 + 2x_2 & & \\ \text{subject to} & & & \\ & -x_2 & \leq & 1 \\ & x_1 & \leq & 1 \\ & -x_1 & \leq & -2 \rightarrow x_1 \geq 2 \\ & x_1 \geq 0. & & \end{array}$$

no feasible solutions since there is no value of x_1 that has $x_1 \leq 1$ and $x_1 \geq 2$